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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

TWENTY-FIFTH SESSION, 1906-1907.

First Meeting, 9th November 1906.

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

For this Session the following Office-Bearers were elected :—

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Vice-President - W. A. LINDSAY, M.A., B.Sc.

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Note on Hypercomplex Numbers.

By J. H. MACLAGAN-WEDDERBURN, M.A.

The present note is an extension of a previous paper* on the same subject. In this paper a concise proof was given of a theorem by Scheffers to the effect that if a linear associative algebra contains the quaternion algebra as a subalgebra, both having the same modulus, then it can be expressed as the direct product of that quaternion algebra and another algebra. It was also shown that this theorem could be generalised to the extent of substituting a matric quadrate algebra for the quaternion algebra. In the present paper the theorem is extended to certain other types of algebras.

As the phrase *linear associative algebra* is rather cumbersome, I use throughout the word *algebra* in its place, and for the same reason I call a subalgebra which has the same modulus as the given algebra a *proper* subalgebra.

1. A number or element of an algebra is said to be rational with respect to a given field or domain if all its coefficients are rational in the same field and the algebra itself is said to be rational if the product of any two rational elements is also rational. Thus Hamilton's quaternions are rational in the field of rational numbers, and hence also in any subfield of the field of real numbers.

Let A be an algebra rational in a given field and having a rational proper subalgebra B which on extending the field is found to be equivalent to the matric quadrate algebra

$$e_{pq}\ (p,\ q=1,\ 2,\dots n\ ;\ e_{pq}e_{qr}=e_{pr},\ e_{pq}e_{rs}=0,\ q\neq r)\ ;$$

then ***the algebra A can be expressed as the direct product of B and some other rational subalgebra.***

By the extension of Scheffers' theorem mentioned in the introduction, A can be expressed in the extended field as the direct product of B and another algebra D. Let the bases of A and B be

$$\mathrm{A}=x_1,\ x_2,\ \dots\ x_a$$
$$\mathrm{B}=y_1,\ y_2,\ \dots\ y_b.$$

These elements can be chosen rational and the proof of the theorem consists in showing that a rational basis can also be chosen for D.

---

* *Proceedings of the Royal Society of Edinburgh*, vol. 26, 1906.

Any element z of D can evidently be expressed in the form

$$z = \Sigma\xi_r x_r' + x'$$

where $x_1', \dots x_s', x'$ are linearly independent rational elements of A and $\xi_1, \dots \xi_s$ are irrational scalars which are linearly independent in the given field. If now y is any rational element of B

$$yz = zy$$

therefore $$0 = \Sigma\xi_r(yx_r' - x_r'y) + yx' - x'y.$$

Hence as the ξ's are independent we must have

$$yx_r' - x_r'y = 0 \quad (r = 1, 2, \dots s)$$
$$yx' - x'y = 0$$

for every rational element y of B and therefore $x_1', x_2', \dots x_s', x'$ are elements of D, *i.e.*, a rational basis can be chosen for D.

2. The second extension is as follows.

If an algebra A contains a rational proper subalgebra B in which (1) *every element has an inverse,* (2) *no element save the modulus is commutative with every other element, then A can be expressed as the direct product of B and a rational algebra C.*

Let $y_1, y_2 \dots y_b$ be a rational basis and

$$y = \Sigma\xi_r y_r$$

any rational element of B. If now we form

$$y' = y_r y - yy_r = \Sigma\xi_s(y_r y_s - y_s y_r) = \Sigma\xi_s y_s'$$

the coefficient ξ_r disappears possibly along with some others. Similarly if ξ_s is any coefficient which has not been eliminated by this process,

$$y'' = y_s'y' - y'y_s' = \Sigma\xi_t y_t''$$

does not contain the coefficient ξ_s and so on. This process may come to an end in two ways.

(1) If in $$y^{(r)} = \Sigma\xi_t y_t^{(r)}$$
$y_1^{(r)}, y_2^{(r)}, \dots$ are all commutative,

(2) If $$y^{(r)} = \xi_t y_t^{(r)}.$$

The first case reduces to the second if $\rho = 1$, so we may suppose $\rho > 1$. Now from the conditions imposed on B, there is an element x which is not commutative with $y_p^{(r)}y_q^{(r)-1}$, unless $y_p^{(r)} = y_q^{(r)}$. But by

replacing $y^{(r-1)}$ by $x'y^{(r)}$ where x' is suitably chosen, we can always arrange that $y_p^{(r)} \neq y_q^{(r)}$, hence we may assume

$$y^{(r)}y_q^{(r)-1}x - xy^{(r)}y_q^{(r)-1} = \Sigma\xi_t(y_t^{(r)}y_q^{(r)-1}x - xy_t^{(r)}y_q^{(r)-1}) \neq 0.$$

The coefficient of ξ_q under the summation sign is, however, zero. Any other coefficient can be eliminated in the same way, and by this process we can reduce the number of terms under the summation sign step by step till an equation of the second type is reached. In both cases, therefore, we are led to an equation of the form

$$y_t = \xi_t y_t^{(r)}$$

or

$$\xi_t = y_t y_t^{(r)-1}$$

where ξ_t is any preassigned coefficient of y. Hence remembering that y is an arbitrary element of B, we see that we can represent its rth coordinate in the form

$$\xi_r = f_r(y)$$

where the form of $f_r(\ )$ does not depend on y. Hence, as was proved in the paper referred to in the introduction, A can be expressed as the direct product of B and the algebra obtained from A by forming

$$f_r(x_s) \quad (r = 1, 2, \dots b\,;\ s = 1, 2, \dots a)$$

x_1, x_2, ... being a basis of A.

Various Questions relative to the Triangle.

By Ed. Collignon.

1. If in a closed area CD a point O is given, and through this point a straight line MN is to be drawn such that the segment MCN so determined may be a minimum or a maximum, this condition is satisfied by drawing through O a chord MN having O for its mid point. Of the two segments which are separated by this chord and which are obtained by rotating the chord round O, the one MDN will be the maximum and the other MCN will be the minimum. The sum of the segments is equal to the whole area of the contour, and consequently if one of them corresponds to the minimum, the complementary segment corresponds to the maximum. The two segments interchange, the one into the other, when the chord MN, turning round the point O, undergoes a displacement equivalent to the angle π.

Apply what has preceded to a triangle ABC, inside which a point O is given.

Through O draw OH parallel to AC and meeting CB at H; and make $Hm = CH$. The straight line mOn, whose mid point is O (Fig. 1), cuts off the minimum triangle mCn (constructed within the angle C), and the complementary area $mBAn$ is the maximum.

In order that the extremities m and n of the chord inscribed in the triangle may be situated on the sides and not on the sides produced, it is necessary and sufficient that the segments Cm and Cn should be respectively less than the sides CB, CA or that their halves CH, HO should be respectively less than the halves of these sides.

If therefore straight lines such as mn, drawn through the point O within the three angles of the triangle ABC, are to be inscribed in the triangle, that is to say, are to have their extremities situated on the sides and not on the sides produced, it is necessary and sufficient to take the point O inside the triangle A′B′C′ whose vertices are the mid points of the sides of triangle ABC.

To simplify the expressions, put

$$\xi = \frac{x}{a} \quad \eta = \frac{y}{b}.$$

The ratios then become

$$\begin{aligned}
&(1)\quad R(A) = 4(\xi - \xi^2 - \xi\eta)\\
&(2)\quad R(B) = 4(\eta - \eta^2 - \xi\eta)\\
&(3)\quad R(C) = 4\xi\eta.
\end{aligned}$$

Between the three equations which we have just obtained eliminate ξ and η. The final equation will be the relation which necessarily exists between the three ratios R(A), R(B), R(C) independently of the form and the dimensions of the triangle as well as the situation of the point O.

To make this elimination, put R(C) for $4\xi\eta$ in equations (1) and (2); then

$$(4)\quad \xi - \xi^2 = \frac{R(A) + R(C)}{4}$$

$$(5)\quad \eta - \eta^2 = \frac{R(B) + R(C)}{4}.$$

From these two equations by addition and by subtracting half of equation (3), we have

$$\begin{aligned}
(6)\qquad \xi + \eta - (\xi^2 + 2\xi\eta + \eta^2) &= (\xi + \eta) - (\xi + \eta)^2\\
&= \frac{R(A) + R(B) + 2R(C)}{4} - \frac{R(C)}{2}\\
&= \frac{R(A) + R(B)}{4}.
\end{aligned}$$

From equations (4) and (5) by multiplication we have

$$\begin{aligned}
(7)\qquad \xi\eta - \xi\eta(\xi + \eta) + \xi^2\eta^2 &= \frac{R(A) + R(C)}{4} \times \frac{R(B) + R(C)}{4}\\
&= \frac{R(A)R(B) + R(C)\{R(A) + R(B)\} + R^2(C)}{16}
\end{aligned}$$

an equation which, if we put $\frac{R(C)}{4}$ for $\xi\eta$, becomes

$$\begin{aligned}
(8)\qquad &\frac{R(C)}{4} - \frac{R(C)}{4}(\xi + \eta) + \frac{R^2(C)}{16}\\
&= \frac{R(A)R(B) + R(C)\{R(A) + R(B)\} + R^2(C)}{16}.
\end{aligned}$$

The terms $\frac{R^2(C)}{16}$ destroy each other, and solving for $\xi+\eta$ we have

$$(9) \qquad \xi+\eta = \frac{\frac{1}{4}R(C) - \frac{1}{16}[R(A)R(B) + R(C)\{R(A)+R(B)\}]}{\frac{1}{4}R(C)}$$

$$= 1 - \frac{R(A)}{4} - \frac{R(B)}{4} - \frac{R(A)R(B)}{4R(C)}.$$

If this value be substituted in equation (6) we have the relation sought:

(10)

$$\left(1 - \frac{R(A)}{4} - \frac{R(B)}{4} - \frac{R(A)R(B)}{4R(C)}\right) - \left(1 - \frac{R(A)}{4} - \frac{R(B)}{4} - \frac{R(A)R(B)}{4R(C)}\right)^2 = \frac{R(A)+R(B)}{4}.$$

Now this relation is not in the definitive form which may be given to it, and which ought to be symmetrical with regard to the three ratios, for the three triangles play the same part in the given triangle.

To reach this form, begin by making symmetrical the parentheses of the first side. We may put

$$(11) \quad R(A)+R(B)+\frac{R(A)R(B)}{R(C)} = \frac{R(B)R(C)+R(C)R(A)+R(A)R(B)}{R(C)}$$

and denote by V the numerator of the second side

$$(12) \qquad V = R(B)R(C)+R(C)R(A)+R(A)R(B)$$

the sum of the products two by two of the three ratios. We shall then have

$$(13) \qquad \left(1-\frac{V}{4R(C)}\right)-\left(1-\frac{V}{4R(C)}\right)^2 = \frac{R(A)+R(B)}{4}.$$

Developing the square and simplifying we have

$$(14) \qquad \frac{V}{4R(C)} - \frac{V^2}{16R^2(C)} = \frac{R(A)+R(B)}{4}.$$

But from equation (11)

$$\frac{V}{4R(C)} = \frac{R(A)+R(B)}{4} + \frac{R(A)R(B)}{4R(C)}.$$

Hence equation (14) reduces to

$$\text{(15)} \qquad \frac{R(A)R(B)}{4R(C)} - \frac{V^2}{16R^2(C)} = 0$$

whence, multiplying by $16R^2(C)$

$$\text{(16)} \qquad 4R(A)R(B)R(C) = V^2$$
$$= \{R(B)R(C) + R(C)R(A) + R(A)R(B)\}^2$$

a formula symmetrical with respect to R(A), R(B), R(C).

Developing the square and dividing by the first side we obtain a form somewhat simpler

(17)

$$\tfrac{1}{4}\left\{\frac{R(B)R(C)}{R(A)} + \frac{R(C)R(A)}{R(B)} + \frac{R(A)R(B)}{R(C)}\right\} + \frac{R(A) + R(B) + R(C)}{2} = 1.$$

Examination of Particular Cases.
Verifications.

4. Place the point O at G the centre of gravity of the triangle.

If through this point we draw parallels to the sides of the triangle, they will have their mid points at G and will satisfy the conditions of the problem.

The three triangles mCn, pAq, rBs are equivalent in this particular case; they are similar to the given triangle; and consequently the ratios R(A), R(B), R(C) are equal, and, the ratio of the sides being $\frac{2}{3}$, the ratio of the surfaces is $\frac{4}{9}$. The final equation (16) or (17) should therefore be satisfied by substituting this value for each of the ratios. We have

form (16) $\quad 4 \times (\frac{4}{9})^3 = \{(\frac{4}{9})^2 \times 3\}^2$

form (17) $\quad \frac{1}{4} \times (\frac{4}{9} \times 3) + \frac{3}{2} \times \frac{4}{9} = 1.$

Take the point O at I the mid point of the side AC.

The finite line AC is, within the angle ABC, the only straight line which is divided at the point I into two equal parts. This line is also the only limiting straight line having its mid point at I, and contained within the two angles ACB, CAB.

Relatively to the first signification of AC we have $R(B) = 1$, since the corresponding triangle is the triangle ABC itself; relatively to the two others we have $R(A) = R(C) = 0$. These values verify the two forms of the general condition.

If we take arbitrarily two ratios R(A), R(B), for example, the equation (16) will give the third ratio R(C). It is of the second degree with respect to the unknown quantity, and consequently furnishes two values for the ratio sought; they will be of the same sign. Choose either of them. We may then construct the triangle ABC, taking arbitrarily the two sides a and b and the angle C contained between them; the ratio R(C) will be related to the triangle constructed within this angle C. We have then, making use of the coordinates ξ and η, referred to the sides a and b

$$\xi+\eta=1-\frac{R(B)R(C)+R(C)R(A)+R(A)R(B)}{4R(C)}$$
$$\xi\eta=\frac{R(C)}{4}.$$

These coordinates will be given by the equation of the second degree

$$t^2-\left\{1-\frac{R(B)R(C)+R(C)R(A)+R(A)R(B)}{4R(C)}\right\}t+\frac{R(C)}{4}=0$$

and may be permuted, the one into the other.

In order that the problem may be possible it is necessary that the values of ξ and η be real; it is necessary besides that the point defined by the coordinates

$$x=a\xi \quad y=b\eta$$

should be within the triangle A′B′C′, which is obtained by joining the mid points of the sides of the given triangle.

We must therefore have the inequalities

$$\xi<\tfrac{1}{2},\ \eta<\tfrac{1}{2},\ \xi+\eta>\tfrac{1}{2}$$

which is equivalent to putting

$$x<\frac{a}{2},\ y<\frac{b}{2},\ bx+ay>\frac{ab}{2}.$$

If these inequalities are not verified, the constructions take place outside the triangle; and the formulæ still apply, but on the condition that suitable signs are given to the areas to be evaluated, according to the conventions of analytical geometry and the integral calculus. The curve passing through the vertices of the hexagon then becomes a hyperbola.

Area of the inscribed hexagon.

5. Let us return to the principal case where (Fig. 2) the point O is contained within the triangle A'B'C', and where the straight lines *mn*, *pq*, *rs* determine the vertices of a hexagon inscribed in the triangle ABC.

Let Ω be the area of this figure.

Its value will be obtained by subtracting from the area S of the triangle ABC, the three triangles A*nr*, B*mp*, C*qs*, formed within the angles A, B, C by the sides parallel to the sides of the given triangle, and which are consequently similar to this triangle. We have therefore

$$\text{area } An r = S \times \left(\frac{An}{AC}\right)^2 = S \times \left(\frac{b-2y}{b}\right)^2$$

$$\text{area } Cqs = S \times \left(\frac{Cq}{CA}\right)^2 = S \times \left(\frac{b-2y'}{b}\right)^2$$

$$\text{area } Bmp = S \times \left(\frac{Bm}{BC}\right)^2 = S \times \left(\frac{a-2x}{a}\right)^2$$

and hence

$$(18) \qquad \Omega = S - S\left\{\left(\frac{b-2y}{b}\right)^2 + \left(\frac{b-2y'}{b}\right)^2 + \left(\frac{a-2x}{a}\right)^2\right\}.$$

In this equation substitute for y' its value in terms of x and y, and for $\frac{x}{a}$, $\frac{y}{b}$ the proportional coordinates ξ and η; then

$$\frac{b-2y}{b} = 1 - 2\eta$$

$$\frac{b-2y'}{b} = \frac{b - \dfrac{2ab-2bx-2ay}{a}}{b}$$

$$= 1 - 2(1-\xi-\eta) = 2(\xi+\eta) - 1$$

$$\frac{a-2x}{a} = 1 - 2\xi$$

and finally

$$(19) \qquad \Omega = S[1-(1-2\eta)^2 - \{2(\xi+\eta)-1\}^2 - (1-2\xi)^2]$$

$$= S[8(\xi+\eta) - 8(\eta^2+\eta\xi+\xi^2) - 2]$$

which may be written

$$(20) \qquad \xi+\eta-(\eta^2+\eta\xi+\xi^2) = \frac{\Omega}{8S} + \tfrac{1}{4}.$$

If the area Ω is given in advance, or rather the ratio $\frac{\Omega}{S}$, the equation (20) defines the curve along which the point O must be taken in order that the corresponding hexagon may have the given area Ω.

To verify the formula take successively

$$\xi = 0 \quad \eta = \tfrac{1}{2}$$
$$\xi = \tfrac{1}{3} \quad \eta = \tfrac{1}{3}.$$

Formula (20) gives for the first hypothesis $\Omega = 0$; and for the second $\Omega = \frac{8S}{12} = \frac{2}{3}S$, which agrees with the results already obtained.

The curve which gives to the area Ω a determinate value is an ellipse

$$\eta^2 + \xi\eta + \xi^2 - \eta - \xi + \tfrac{1}{4} = \text{constant}.$$

It has for centre the point

$$\xi = \tfrac{1}{3} \quad \eta = \tfrac{1}{3}$$

that is to say, the centre of gravity of the triangle.

If we vary the constant, all the curves are similar and have the point G, the centre of gravity of the triangle, for centre of similitude.

If we wish to know the maximum value of the area Ω, it is sufficient to put equal to zero the partial derivatives of the function which represents the value of $\frac{\Omega}{8S}$. This gives

$$2\eta + \xi - 1 = 0$$
$$\eta + 2\xi - 1 = 0$$

that is, $\xi = \eta = \frac{1}{3}$, coordinates of the centre of gravity.

The hexagon is therefore a maximum for the point G, the common centre of the ellipses. At this point we have $\Omega = \frac{2}{3}S$, and the ellipse reduces to its centre.

If we put $\Omega = 0$, the equation of the curve becomes

$$\eta^2 + \xi^2 + \xi\eta - \eta - \xi + \tfrac{1}{4} = 0$$

which represents the ellipse passing through the mid points of the three sides and having these sides for tangents.

A similar ellipse can be inscribed in the triangle A′B′C′. One of the points of contact has for coordinates $\xi = \eta = \frac{1}{4}$, and consequently for all the points of this ellipse we shall have $\Omega = \frac{1}{2}S$.

Among these similar ellipses must be counted that which passes through the three vertices A, B, C. If we take the value of Ω for the vertex C as origin, we shall have

$$\xi = \eta = 0$$

and consequently $\dfrac{\Omega}{8S} + \frac{1}{4} = 0$

whence $\Omega = -2S$.

We can express the area Ω as a function of the ratios $R(A), R(B), R(C)$.

For we have

$$\xi\eta = \frac{R(C)}{4}$$

$$\xi + \eta = 1 - \tfrac{1}{4}\left\{R(A) + R(B) + \frac{R(A)R(B)}{R(C)}\right\}$$

$$\eta^2 + \xi\eta + \xi^2 = (\eta + \xi)^2 - \xi\eta$$

$$= \left[1 - \tfrac{1}{4}\left\{R(A) + R(B) + \frac{R(A)R(B)}{R(C)}\right\}\right]^2 - \frac{R(C)}{4}$$

and equation (20) becomes

$$1 - \tfrac{1}{4}\left\{R(A) + R(B) + \frac{R(A)R(B)}{R(C)}\right\}$$

$$- \left[1 - \tfrac{1}{4}\left\{R(A) + R(B) + \frac{R(A)R(B)}{R(C)}\right\}\right]^2 + \frac{R(C)}{4} = \frac{\Omega}{8S} + \tfrac{1}{4}$$

Since we have transformed the first side into $\dfrac{R(A) + R(B)}{4}$, by equation (13) we have

$$\frac{\Omega}{8S} + \tfrac{1}{4} = \frac{R(A) + R(B) + R(C)}{4}$$

that is

$$\Omega = 8S\left[\frac{R(A) + R(B) + R(C)}{4} - \tfrac{1}{4}\right]$$

$$= 2S\{R(A) + R(B) + R(C) - 1\}.$$

Second Meeting, 14th December 1906.

J. ARCHIBALD, Esq., M.A., President, in the Chair.

A Problem in Conics.

By R. F. DAVIS, M.A.

The locus of the focus of a parabola touching three given fixed straight lines is both exceedingly simple and widely known. On the other hand the analogous locus of the focus of a parabola passing through three given fixed points is excessively complicated, and its investigation has, so far as the present writer knows, never appeared in any text-book.

The analysis for the general case is so cumbrous that it would appear the easiest course to deal first with the simplest case and then gradually build up and generalise.

Let Ox, Oy be rectangular axes upon which fixed points A, B are respectively taken ($OA = OB = 2a$). If p, q, r are perpendiculars from O, A, B (Fig. 3) upon the directrix of a parabola passing through OAB, then*

$$\frac{(q-p)^2}{4a^2} + \frac{(r-p)^2}{4a^2} = 1$$

or

$$(q-p)^2 + (r-p)^2 = 4a^2.$$

Hence if S is the focus, since $SP = PM$ in the parabola,

$$(SA - SO)^2 + (SB - SO)^2 = 4a^2,$$

which may be regarded as a sort of "tripolar" equation to the locus of S.

* This relation is of course true for any straight line.

For want of courage I should never have proceeded to reduce this equation to Cartesian coordinates. It has, however, been done very kindly for me by Professor W. H. H. Hudson, of King's College, London: and I have verified the result, which is as follows:

$$(2s-3a)r^4-2s(s-a)^2r^2+a(s-a)^4=0$$

where $s=x+y$, $r^2=x^2+y^2$. The result is therefore a quintic having the lines $x=\frac{a}{2}$, $y=\frac{a}{2}$, $x+y-\frac{3a}{2}=0$ as asymptotes.

Mr C. E. Youngman has investigated, by trilinears, the case where the three given fixed points form an equilateral triangle in the present (November) number of the *Educational Times*. He also finds the locus to be a quintic having three real asymptotes parallel to the sides of the original triangle.

The question I should like to put is—Why does the locus come out of the fifth degree? Is there any geometrical view of the question which could account for this?

The foregoing result has been amply verified. It may be derived

(i) By reducing

$$(SA-SO)^2+(SB-SO)^2=4a^2$$

to Cartesians.

(ii) By eliminating λ from

$$2x/a=(\lambda^4+2\lambda\ -1)/\ (\lambda-1)(\lambda^2+1)$$
$$2y/a=(\lambda^4-2\lambda^3-1)/\lambda(\lambda-1)(\lambda^2+1).$$

(iii) By identifying

$$(x+\lambda y)^2-2ax-2\lambda^2ay=0$$

with

$$(x-\xi)^2+(y-\eta)^2=(\lambda x-y-\kappa)^2/(\lambda^2+1)$$

and then eliminating λ, κ from

$$(\lambda^2+1)\,\xi\ -\lambda\kappa\ =a$$
$$(\lambda^2+1)\,\eta\ +\kappa\ \ =\lambda^2a$$
$$(\lambda^2+1)(\xi^2+\eta^2)\ =\kappa^2.$$

These eliminations are none too easy: and analysis seems powerless to throw any light upon the cause for this particular locus being of such a high degree.

In studying this locus, I was led by analysis to the following theorem, which I afterwards verified geometrically:—

Let a variable parabola be described through three given fixed points E_1, E_2, E_3, and let a tangent triangle $t_1t_2t_3$ be drawn whose sides are parallel to, and one fourth the length of, the sides of $E_1E_2E_3$. Then the locus of t_1 is a cubic hyperbola (locus of a point the product of whose perpendiculars upon three fixed straight lines is constant).

The triangle $t_1t_2t_3$ being of fixed size and shape and moving without rotation, any of its special points (such as its circum-centre) will describe the same curve as t_1 shifted through a constant space in a certain direction.

Thus the circum-circle of $t_1t_2t_3$ has a constant radius and its centre moves on a cubic hyperbola. All that can be said at present is that the focus S lies *somewhere* on this circle. The fact, however, that the quintic curve, which is the locus of S, has three asymptotes coincident with the three asymptotes of the cubic hyperbola, which is the locus of the circum-centre of $t_1t_2t_3$, would seem to show that a geometrical solution (if it is ever to be found) will in some way proceed from the association of these two loci.

De la Vallée Poussin's Extension of Poisson's Integral.

By George A. Gibson, M.A., LL.D.

The integrals of §§ 5, 6, and 7 of the following paper were first established by C. de la Vallée Poussin in a memoir *Sur quelques applications de l'intégrale de Poisson* (*Ann. de la Soc. sc. de Bruxelles*, vol. 17, 1892–3). An analogous integral to that of § 5 was discovered by A. Hurwitz, who seems not to have been aware of de la Vallée Poussin's memoir, and will be found under the title *Sur quelques applications des series de Fourier* in the *Annales de l'École normale*, vol. 19, 1902. In view of the value of these integrals for the theory of the Fourier series, the discussion now given, which follows different lines from those of previous proofs, may be of some interest. The discussion turns chiefly on the Second Theorem of Mean Value which is quite as applicable to Poisson's as to Dirichlet's Integral.

§ 1. Let $f(x)$ be given from $x = -\pi$ to $x = \pi$, and let it be defined for other values of x by the equation $f(x) = f(x \pm 2n\pi)$ where n is a positive integer such that $-\pi < x \pm 2n\pi \leqq \pi$. Suppose $f(x)$ to be either finite and integrable or else, when not finite, to be absolutely integrable, the number of discontinuities in the latter case being limited.

The integral, in which $0 < r < 1$,

$$\phi(r, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(u)(1 - r^2)du}{1 - 2r\cos(u - x) + r^2} \qquad \text{(1)}$$

is a continous function of x for any given r less than 1; we wish to discuss some theorems involving the limit of $\phi(r, x)$ for r converging to unity.

Since $f(u)$ is periodic we may, if convenient, write

$$\phi(r, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(u + x)(1 - r^2)du}{1 - 2r\cos u + r^2} \qquad (1_a)$$

For brevity we shall sometimes put

$$(1 - r^2)/(1 - 2r\cos(u - x) + r^2) = g(r, u - x).$$

§ 2. Let h be any arbitrarily small but fixed positive number and consider the integrals

$$U(r, x) = \frac{1}{2\pi}\int_{x+h}^{\pi} f(u)g(r, u-x)du, \quad V(r, x) = \frac{1}{2\pi}\int_{-\pi}^{x-h} f(u)g(r, u-x)du,$$

$$W(r) = \frac{1}{2\pi}\int_{-\pi+h}^{\pi-h} f(u)\frac{1-r^2}{1+2r\cos u + r^2}du.$$

In the integrals $U(r, x)$ $V(r, x)$ x may have any value in the range $(-\pi+h, \pi-h)$ and the sum $U+V$ gives for such x the value of the integral in equation (1) except for the contribution arising from the range from $u=x-h$ to $u=x+h$ which constitutes the neighbourhood of x. The neighbourhood of the points $x=\pm\pi$ is to be considered as made up of the two parts from $-\pi$ to $-\pi+h$ and from $\pi-h$ to π; the integral $W(r)$ gives the value of the integral in equation (1) for $x=\pm\pi$ except for the contribution arising from the neighbourhood of these points.

We show that as r tends to unity $W(r)$ tends to zero and $U(r, x)$ and $V(r, x)$ tend uniformly to zero.

Take $U(r, x)$, which by putting $u+x$ for u takes the form

$$U(r, x) = \frac{1}{2\pi}\int_{h}^{\pi-x} f(u+x)\frac{1-r^2}{1-2r\cos u + r^2}du.$$

The function $(1-r^2)/(1-2r\cos u+r^2)$ or $g(r, u)$ is positive and decreasing from $u=0$ to $u=\pi$ but positive and increasing from $u=\pi$ to $u=2\pi$; the second theorem of mean value can therefore be used. If $0 \leqq x \leqq \pi-h$ the function $g(r, u)$ is positive and decreasing within the range of integration and only one application of the theorem is needed; if $-\pi+h \leqq x < 0$, then x is negative and $\pi-x$ is greater than π so that two applications are needed.

Suppose x to be negative; then

$U(r, x)$

$$= \frac{1}{2\pi}\int_{h}^{\pi} f(u+x)\frac{1-r^2}{1-2r\cos u + r^2}du + \frac{1}{2\pi}\int_{\pi}^{\pi-x} f(u+x)\frac{1-r^2}{1-2r\cos u + r^2}du$$

$$= \alpha + \beta, \text{ say,}$$

$$\alpha = \frac{1-r^2}{2\pi(1-2r\cosh h + r^2)}\int_{h}^{\xi} f(u+x)du, \quad h \leqq \xi \leqq \pi$$

since $g(r, u)$ is in this case a positive decreasing function;

$$\beta = \frac{1-r^2}{2\pi(1+2r\cos x + r^2)}\int_{\xi'}^{\pi-x} f(u+x)du, \quad \pi \leqq \xi' \leqq \pi - x$$

since $g(r, u)$ is now a positive increasing function.

The integrals

$$\int_h^{\xi} f(u-x)du, \int_{\xi'}^{\pi-x} f(u+x)dx$$

are each numerically less than the integral

$$\int_{-\pi}^{\pi} |f(u)|\, du$$

which is finite and independent of x.

The factor $(1-r^2)/2\pi(1-2r\cos h+r^2)$ converges to zero as r converges to unity since h is not zero. (It is easy to see that if h is not fixed but tends to zero as r tends to unity the limit is still zero provided that the limit of $(1-r)/h^2$ is zero when r tends to unity.) The factor $(1-r^2)/2\pi(1+2r\cos x+r^2)$ is not greater than $(1-r^2)/2\pi(1+2r\cos h+r^2)$ and therefore tends to zero as r tends to unity whatever value x may have within the range to which it is confined.

Thus $U(r, x)$ tends uniformly to zero as r converges to unity. Similar considerations hold for the integrals $V(r, x)$ and $W(r)$.

We have therefore the result that if h is any fixed positive number, no matter how small if it be different from zero, and x any number such that $-\pi+h \leqq x \leqq \pi-h$, it is always possible to take r so near to unity that the sum

$$| U(r, x) + V(r, x) |$$

will become and, as r is taken closer to unity, will remain less than any given arbitrarily small positive number.

§ 3. Let us suppose for the present that $f(x)$ is finite and integrable, the case of infinite discontinuities being considered later. Make a division of the range from $-\pi$ to π and consider those intervals, e_s say, in which the oscillation of $f(x)$ exceeds σ, where σ is any arbitrarily small positive number. Since $f(x)$ is integrable it must be possible to choose the intervals e_s so that Σe_s will be less than any given arbitrarily small positive number ϵ; let σ, ϵ be given and the intervals e_s determined.

Next enclose each interval e_s in an interval e_s' so that e_s' overlaps e_s at both ends and let $\Sigma e_s'$ be less than 2ϵ; denote by A' what is left of the range $(-\pi, \pi)$ when the intervals e_s' have been removed. The extremities $-\pi$ and π will not usually belong to A' but must be assigned to e_s'.

We now choose the number h of § 2; we take h to be any fixed number less than the least distance from an extremity of e_s to the corresponding extremity of the e_s' which overlaps it. When h is so chosen we shall have

$$|f(x+u)-f(x)| \leqq \sigma \text{ if } 0 \leqq |u| \leqq h$$

for every x in A′.

Now let x be any number in A′.

$$\phi(r, x) = \mathrm{U}(r, x) + \mathrm{V}(r, x) + \frac{1}{2\pi}\int_{x-h}^{x+h} f(u)g(r, u-x)du.$$

The last term in this equation may be written

$$\frac{1}{2\pi}\int_{x}^{x+h} f(u)g(r, u-x)du + \frac{1}{2\pi}\int_{x-h}^{x} f(u)g(r, u-x)du$$

$$= \mathrm{I}_1 + \mathrm{I}_2, \text{ say.}$$

$$\mathrm{I}_1 = \frac{1}{2\pi}\int_0^h f(x+u)g(r, u)du$$

$$= \frac{f(x)}{2\pi}\int_0^h g(r, u)du + \frac{1}{2\pi}\int_0^h \{f(x+u)-f(x)\}g(r, u)du.$$

By taking r close enough to unity the first term may be made to differ as little as we please from $\frac{1}{2}f(x)$. Next

$$\frac{1}{2\pi}\left|\int_0^h \{f(x+u)-f(x)\}g(r, u)du\right| \leqq \frac{\sigma}{2\pi}\int_0^h g(r, u)du$$

and therefore $\leqq \frac{1}{2}\sigma$

whatever value less than unity is assigned to r. Thus we can choose ρ' so that when $\rho' < r < 1$

$$|\mathrm{I}_1 - \tfrac{1}{2}f(x)| < \tfrac{1}{2}\sigma + \tfrac{1}{2}\sigma, \text{ that is, } < \sigma;$$

and, in exactly the same way, it follows that when $\rho'' < r < 1$

$$|\mathrm{I}_2 - \tfrac{1}{2}f(x)| < \sigma.$$

By § 2, ρ''' can be chosen so that when $\rho''' < r < 1$ the sum $|\mathrm{U}(r, x) + \mathrm{V}(r, x)|$ will be as small as we please, say less than σ.

Thus we have shown that if x is any number in A′ we can choose a number ρ such that when $\rho < r < 1$

$$|\phi(r, x) - f(x)| < 3\sigma;$$

in other words, when x is any number in A′, $\phi(r, x)$ converges uniformly to $f(x)$.

If x is not in A′ let M be an upper limit to the values of $f(x)$; then

$$|\phi(r, x)| < \frac{M}{2\pi}\int_{-\pi}^{\pi} g(r, u-x)du \leqq M$$

so that $|\phi(r, x)|$ is finite for every r.

It is important to observe that, whether $f(x)$ has infinite discontinuities or not, $|I_1 + I_2|$ is finite for every r less than unity when x is not in the neighbourhood of a point of infinite discontinuity. For if M is an upper limit to the values of $|f(u)|$ from $u = x - h$ to $u = x + h$

$$|I_1 + I_2| < \frac{M}{2\pi}\int_{x-h}^{x+h} g(r, u-x)du \leqq M.$$

Hence as r tends to unity $|\phi(r, x)|$ remains finite, because $|U(r, x) + V(r, x)|$ may be made as small as we please.

§ 4. Let $F(x)$ satisfy the conditions stated for $f(x)$ in § 1; when $F(x)$ is finite and integrable from $x = -\pi$ to $x = \pi$ the function $\psi(r, x)$, defined by the equation

$$\psi(r, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(v)g(r, v-x)dv$$

converges uniformly to $F(x)$, when r tends to unity, for every number x in the range A″, corresponding to A′, and in the other parts of the range is finite, less than or equal to the upper limit of $|F(x)|$.

§ 5. When $f(x)$ and $F(x)$ are finite and integrable

$$\underset{r=1}{L}\int_a^b \phi(r, x)\,\psi(r, x)dx = \int_a^b f(x)\,F(x)dx \quad - \quad - \quad (1)$$

where $-\pi \leqq a < b \leqq \pi$.

Denote by B′ what is left of the range (a, b) after removal of the intervals e_s', determined now for both functions $f(x)$ and $F(x)$. If x is any number in B′ the product $\phi(r, x)\psi(r, x)$ converges uniformly to the product $f(x)F(x)$ and therefore

$$\underset{r=1}{L}\int_{B'} \phi(r, x)\,\psi(r, x)dx = \int_{B'} f(x)\,F(x)dx \quad - \quad - \quad (2)$$

the integration extending over the range B′, that is, from a to b but excluding the invervals e_s'.

Suppose $|f(x)|$ and $|F(x)|$ each less than M for every x not in B′; then the product $|\phi(r, x)\,\psi(r, x)|$ is less than M^2 for every such x. Hence

$$\left|\int_{E'} \phi(r, x)\,\psi(r, x)dx\right| < M^2\Sigma e_s'\,;\ \left|\int_{E'} f(x)\,F(x)dx\right| < M^2\Sigma e_s$$

the integration extending over all the intervals e_s'. But $\Sigma e_s'$ can be made as small as we please, so that equation (2) becomes.

$$\underset{r=1}{L}\int_a^b \phi(r, x)\,\psi(r, x)dx = \int_a^b f(x)\,F(x)dx.$$

§ 6. Suppose now that $f(x)$ and $F(x)$ have infinite discontinuities; it will be sufficient to suppose $f(x)$ infinite for $x = c$ and $F(x)$ infinite for $x = c'$. The functions are assumed to be absolutely integrable. When c and c' are unequal the product $f(x)F(x)$ will also be absolutely integrable; but when c and c' are equal it does not necessarily follow that the product is absolutely integrable so that when $c = c'$ the express hypothesis is made that the product is absolutely integrable. The theorem of § 5 holds in both cases, as will now be seen.

Equation (2) of § 5 is still true provided B′ denote the range (a, b) with the neighbourhoods of c and c' removed. To establish the theorem therefore we have merely to show that the contribution to the integrals of equation (1) § 5 from values of x in the neighbourhood of c and c' can be made as small as we please by diminishing the extent of the neighbourhood.

First suppose c and c' unequal. The integrals

$$\left|\int_{c-\eta}^{c+\eta'} f(x)\,F(x)dx\right|,\quad \left|\int_{c'-\eta}^{c'+\eta} f(x)\,F(x)dx\right|$$

may by choice of η and η' be made arbitrarily small, because the product $f(x)F(x)$ is absolutely integrable.

When x is in the neighbourhood of c, $|\psi(r, x)|$ is less than a finite number M, and

$$\begin{aligned}\phi(r, x) &= \frac{1}{2\pi}\int_{x-h}^{x+h} f(u)g(r, u-x)du + \gamma \\ &= I + \gamma,\ \text{say,}\end{aligned}$$

where γ tends uniformly to zero as r tends to unity. Now

$$\left|\int_{c-\eta}^{c+\eta'} \mathrm{I}\psi(r, x)dx\right| \leqq \frac{\mathrm{M}}{2\pi}\left|\int_{c-\eta}^{c+\eta'} dx \int_{-h}^{h} f(x+u)g(r, u)du\right|$$

$$\leqq \frac{\mathrm{M}}{2\pi}\left|\int_{-h}^{h} g(r, u)du \int_{c-\eta}^{c+\eta'} f(x+u)dx\right|$$

$$< \mathrm{M}\epsilon$$

by proper choice of η and η', because $f(x+u)$ is absolutely integrable.

Next

$$\left|\int_{c-\eta}^{c+\eta'} \gamma\psi(r, x)dx\right| < \mathrm{M}\,|\gamma_1|\,(\eta+\eta')$$

where $|\gamma_1|$ is an upper limit to $|\gamma|$.

Thus

$$\left|\int_{c-\eta}^{c+\eta'} \phi(r, x)\,\psi(r, x)dx\right| = \left|\int_{c-\eta}^{c+\eta'} (\mathrm{I}+\gamma)\,\psi(r, x)dx\right|$$

$$< \mathrm{M}\epsilon + \mathrm{M}\,|\gamma_1|(\eta+\eta')$$

and can therefore be made arbitrarily small.

A similar investigation is applicable when x is in the neighbourhood of c', so that the theorem of § 5 still stands.

Next, suppose c and c' to be equal. The integral

$$\left|\int_{c-\eta}^{c+\eta'} f(x)\,\mathrm{F}(x)dx\right|$$

may be made arbitrarily small by choice of η and η'.

We also need the result that if $-h \leqq u \leqq h$, $-h \leqq v \leqq h$ the integral

$$\left|\int_{c-\eta}^{c+\eta'} f(x+u)\,\mathrm{F}(x+v)dx\right|$$

may be made arbitrarily small (less than ϵ) by choice of η, η'. Whether u and v are equal or not the product $f(x+u)\,\mathrm{F}(x+v)$ is absolutely integrable and the result is evident.

If we write

$$\psi(r, x) = \frac{1}{2\pi}\int_{x-h}^{x+h} \mathrm{F}(v)g(r, v-x) + \delta = \mathrm{J} + \delta$$

and put $\mathrm{I}+\gamma$ for $\phi(r, x)$ as before, we may obviously disregard γ and δ, as a process precisely similar to that just used will show.

We have therefore to examine the integral of I × J.

$$\int_{c-\eta}^{c+\eta'} \mathrm{IJ}dx = \frac{1}{4\pi^2}\int_{c-\eta}^{c+\eta'} dx \int_{-h}^{h} f(x+u)g(r, u)du \int_{-h}^{h} \mathrm{F}(x+v)g(r, v)dv$$

$$= \frac{1}{4\pi^2}\int_{-h}^{h} g(r, u)du \int_{-h}^{h} g(r, v)dv \int_{c-\eta}^{c+\eta'} f(x+u)\mathrm{F}(x+v)dx.$$

Thus

$$\left|\int_{c-\eta}^{c+\eta} \mathrm{IJ}dx\right| < \frac{\epsilon}{4\pi^2}\int_{-h}^{h} g(r, u)du \int_{-h}^{h} g(r, v)dv$$

and therefore less than ϵ.

The theorem of § 5 is thus true under the conditions stated. Of course if there were more than one infinity, the process would be repeated for each, so that the theorem holds when the number of points of infinite discontinuity is finite.

§ 7. If $\chi(x)$ is finite and integrable, the more general equation holds, namely,

$$\underset{r=1}{\mathrm{L}} \int_a^b \phi(r, x)\psi(r, x)\chi(x)dx = \int_a^b f(x)\,\mathrm{F}(x)\,\chi(x)dx$$

as is very readily seen after what has been done.

Points at Infinity, etc., in a plane.

By P. PINKERTON, M.A.

1. The object of this paper is to correlate the geometrical and analytical aspects of the elements of the theory of points at infinity, etc., in a plane. It is assumed that the reader is acquainted with the method of tracing the graphs of rational functions of x, by using the artifices of change of origin and approximation by Ascending or Descending Division (see Chrystal's *Introduction to Algebra*, Ch. XXV.).

2. When we write $\tan\frac{\pi}{2} = \infty$, geometrically we mean (1) that a right-angled triangle can be constructed, the ratio of whose sides is equal to n, where n is any pre-assigned positive number, however large, and (2) that the difference between the greater of the acute angles of the triangle and a right angle continually tends to zero as n increases. The full notation is $\underset{\theta=\frac{\pi}{2}}{\text{Lt.}} \tan\theta = \infty$. Algebraically, we have $\tan\frac{\pi}{2} = r \div 0$, where r is the length of the radius vector which, by revolving according to the general definition of the tangent of an angle, traces out the angle $\frac{\pi}{2}$ radians. But $r \div 0$ is not a particular or special case of the algebraic operations which obey the Laws of Algebra; for, if it were, we might reason thus:

$0 \times 8 = 0$ and $0 \times 9 = 0$, $\therefore\ 0 \times 8 = 0 \times 9$, $\therefore\ 0 \times 8 \div 0 = 0 \times 9 \div 0$, $\therefore\ 8 \times (0 \div 0) = 9 \times (0 \div 0)$, $\therefore\ 8 = 9$. Thus $\tan\frac{\pi}{2}$ has no algebraic value, *i.e.*, is not equal to a number which obeys the Laws of Algebra. We call $r \div 0$ a *limiting case* of an algebraic operation, and write, *in this case*, $r \div 0 = \infty$, the full notation being again $r \div 0 = \underset{x=0}{\text{L}}\, r \div x = \infty$. Again a tangent to a circle is a *limiting case* of a secant of a circle; it is not a secant, for it lacks some of the properties of a secant. Correspondingly in analysis; if x_1y_1 and x_2y_2

are the coordinates of the points in which a secant cuts the circle $x^2+y^2=a^2$, then the secant has for equation

$$(y-y_1)/(y_1-y_2)=(x-x_1)/(x_1-x_2).$$

When x_2y_2 coincides with x_1y_1, we get the equation

$$(y-y_1)\div 0=(x-x_1)\div 0,$$

involving limiting cases of algebraic operation; and we have to "evaluate" the equation

$$\underset{y_2=y_1}{\mathrm{L}}\frac{y-y_1}{y_1-y_2}=\underset{x_2=x_1}{\mathrm{L}}\frac{x-x_1}{x_1-x_2},$$

under the conditions $x_1^2+y_1^2=x_2^2+y_2^2=a^2$, to find the equation to the tangent at x_1y_1. The introduction of limiting cases into Geometry finds its equivalent in the introduction of limiting cases of algebraic operation into Analysis.

3. Let A, B be two fixed points on a given straight line, and let t be a variable number, finite both ways, and positive or negative. Through A and B draw AX, BY parallel straight lines such that $AX/BY=t$, account being taken of the directions of AX, BY as well as of their magnitudes (AX, BY, in fact, being *steps*). The line XY meets the line AB unless $t=+1$. Let P be the variable point of intersection; then, clearly, $AP/BP=AX/BY=t$, AP, BP being *steps*. Also if it be supposed possible to find another point Q such that $AQ/BQ=t$, we shall have $AP/BP=AQ/BQ$, $\therefore (AP-BP)/BP=(AQ-BQ)/BQ$, $\therefore AB/BP=AB/BQ$, $\therefore BP=BQ$; so that Q is the same point as P, BP and BQ being steps. Hence *one and only one* point P can be found such that $AP/BP=t$, provided $t\neq+1$. The nearer t is to $+1$ the further is P from A (or B). To make the possible values of t complete, we introduce *as a convention the limiting case* of a point on the line, corresponding to $t=+1$. Since in the case of actual points on the line, there is one and only one point for each value of t, so we make the convention that there is only one point on the line whose "position-ratio" is equal to $+1$. This point is called *the point at infinity on the line*. It is to be observed that the point at infinity on a straight line is not a point in sober fact on the line, any more than a tangent to a circle is a secant. Indeed we have not defined the point at infinity on a line to have position, but only a "position-ratio," the value of the "position-ratio," viz., $+1$, corresponding to *no* position on the line.

Analytically, let B be taken as an origin of coordinates on the line AB, and let the coordinates of A and P be a and x respectively. Then if AP/BP $= t$, we have, to determine x, the equation $\dfrac{x-a}{x} = t$, so that $x = \dfrac{a}{1-t}$, if $t \neq 1$. If $t = 1$, there is *no value* of x; we have the anomalous equation $0 \,.\, x = a$. But making use of limiting cases of algebraic operation, we write $x = a \div 0 = \infty$. If, then, points at infinity are in question, we cannot omit from an equation such a term as mx when $m = 0$, but only when actual (finite) points are in question.

4. We now proceed to endow points at infinity with geometrical and analytical properties, taking care that the process involves us in no contradictions. Geometrically, we may suppose the point at infinity on the line AB joined to a finite point in the plane not having its position on AB. This line cannot meet the line AB again; for we must escape the contradiction that two straight lines have more than one point in common. The straight line must therefore be considered as parallel to AB. A system of such straight lines will be a system of parallel straight lines; that is, parallel straight lines have their respective points at infinity in common, or meet at infinity, as is the usual expression. This result also flows from the geometrical construction of § 3. Analytically: let one of a system of parallel straight lines be chosen as the x-axis of a Cartesian system of reference. Then the equation of any other may be written in the form $y = 0 \,.\, x + c$; the term $0 \,.\, x$ not being omitted, as points at infinity are in question (see § 3). To find the point of intersection of $y = 0 \,.\, x + c$, and the x-axis, we solve the equations $y = 0 \,.\, x$ and $y = 0 \,.\, x + c$, whence arises the anomalous equation $0 \,.\, x + c = 0$, showing (§ 3) that the point of intersection is the point at infinity on the x-axis. Therefore, a system of parallel straight lines "meet at infinity."

5. Associated with any system of parallel straight lines, therefore associated with any direction in a plane, is a point at infinity. Hence associated with a plane is an assemblage of points at infinity. This assemblage of points is met by any straight line in the plane in one and only one point, and therefore is to be regarded as forming a

straight line, called the *line at infinity* in the plane. Referred to a Cartesian system of reference in the plane, the equation to the line at infinity must be written $0 \,.\, x + 0 \,.\, y + c = 0$. For an equation of the first degree is in question; this equation is satisfied by the coordinates of no actual point in the plane, and the point-pair common to $0.x + 0.y + c = 0$ and $Ax + By + C = 0$ is $(\infty, \infty), (\infty, 0)$ or $(0, \infty)$, according as $A \neq 0$ and $B \neq 0$, or $A = 0$ and $B \neq 0$, or $A \neq 0$ and $B = 0$.

6. The necessity of preserving terms like $0 \,.\, x$ in equations, or of understanding that they are present if not actually written, where points at infinity or infinite values of x are in question, may be illustrated geometrically and analytically by considering the coaxal system of circles determined by the equations

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$\text{and} \quad S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

The equation, $S + kS' = 0$, represents the system of coaxal circles except when $k = -1$, when $S + kS' = 0$ takes the form

$$0 \,.\, x^2 + 0 \,.\, y^2 + 2(g - g')x + 2(f - f')y + c - c' = 0,$$

which may be written

$$(0 \,.\, x + 0 \,.\, y + 1)\{2(g - g')x + 2(f - f')y + c - c'\} = 0,$$

representing the line at infinity as well as the line given by

$$2(g - g')x + 2(f - f')y + c - c' = 0.$$

This *limiting case* of the system of circles, $S + kS' = 0$, thus appears as *two* straight lines, one lying wholly at infinity, the other being (as usually defined) the radical axis of the coaxal system defined by the circles $S = 0$, $S' = 0$. Now if a secant through a point P cuts a circle in A and B, the power of the point P with respect to the circle is geometrically defined as PA . PB. When the point P lies on the circle PA . PB $= 0$. But if the radical axis with the line at infinity be included, as a limiting case, in the system of coaxal circles defined by $S = 0$, $S' = 0$, and if P be taken on the radical axis PA . PB assumes the form $0 \times \infty$—an indeterminate form, as is geometrically obvious. For since P lies on the radical axis PA . PB in the limiting case remains equal to the square on the tangent from P to any circle of the system, and this is indeterminate, depending on the position of P.

7. The usefulness of the conventions, point at infinity and line at infinity, in connection with Menelaus' and Ceva's Theorems, Harmonic Ranges and Pencils, etc., is well known. The task of making what is found in text-books clear and intelligible may be left to the reader. The rest of this paper will be occupied with remarks on points at infinity on plane curves.

8. It has already been seen that, when an equation of the first degree takes the anomalous form $0 . x + c = 0$, it is to be interpreted as having its root infinite. If the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ .. + a_rx^{n-r} + \ldots + a_{n-1}x + a_n = 0, \quad - \quad (1)$$

be transformed by putting $x = 1/y$, we obtain the equation

$$a_ny^n + a_{n-1}y^{n-1} + \ldots + a_ry^r + \ldots + a_2y^2 + a_1y + a_0 = 0. \quad - \quad (2)$$

Consider the coefficients a_0, a_1, etc., as varying according to some law; one root of (2) tends to zero as a_0 tends to zero, provided $a_1 \neq 0$; two roots of (2) tend to zero if a_0 and a_1 tend to zero while a_2 does not tend to zero; r roots tend to zero as $a_0, a_1, \ldots, a_{r-1}$ tend to zero while a_r does not tend to zero. But while y tends to zero, x tends to ∞, if we introduce limiting cases of operation, as we have seen we must do if points at infinity and infinite values of x are in question. Hence if $a_0 = 0$ and $a_1 \neq 0$, we make the convention that *one* root of (1) is infinite; if $a_0 = 0$ and $a_1 = 0$ while $a_2 \neq 0$, *two* roots of (1) are to be considered infinite; and so on.

9. These analytical conventions may be graphically illustrated. The case of the equation $0 . x + c = 0$ has its equivalent in the geometrical convention that two parallel straight lines have their points at infinity in common. The cases $0 . x^2 + px + q = 0 (p \neq 0)$, and $0 . x^2 + 0 . x + q = 0 (q \neq 0)$, may be illustrated from the graph of $y = (x-1)/(x+1)$, fig. 4. The quadratic equation

$$(mx + c)(x + 1) = x - 1 \quad - \quad - \quad - \quad - \quad (3)$$

determines the abscissae of the points of intersection of the graph and the line $y = mx + c$. There are in general two such points. When $m = 0$ and $c \neq 1$, equation (3) takes the form $0 . x^2 + c(x+1) = x - 1$; there is one finite point of intersection, whose abscissa is determined by the equation $c(x+1) = x - 1$, and one point of intersection at infinity. The branches of the graph that stretch to right and left tend to assume the form of a parallel to $y = c (c \neq 1)$, consistently

with $\underset{x=\infty}{\mathrm{L}}(x-1)/(x+1)=1$. The abscissae of the points of intersection of $y=1$ and the curve are given by the equation $0 \, . \, x^2+0 \, . \, x+2=0$. The straight line $y=1$ never meets the curve at any actual points. But we are at liberty to say that it meets the curve at two coincident points—these points coinciding with the point at infinity on the line. We thus have a point at infinity on a curve of the second degree behaving like an actual point of simple concavity towards a straight line as tangent. We thus speak of a point of simple concavity at infinity and call the "tangent" at the point an *asymptote*.

10. Curves of higher degree than the second present similar properties. Consider the graph of $y=(-x^2+3x+2)/(x^2+x+1)$, fig. 5. The abscissae of the points of intersection of the straight line $y=mx+c$ with the curve are given by the equation

$$(mx+c)(x^2+x+1)+x^2-3x-2=0. \qquad \text{-} \quad \text{-} \quad (4)$$

A cubic equation is in question. When the straight line $y=mx+c$ is parallel to the x-axis, $m=0$; and equation (4) takes the limiting form

$$0 \, . \, x^3+(c+1)x^2+(c-3)x+(c-2)=0. \qquad \text{-} \quad \text{-} \quad (5)$$

If $c+1 \neq 0$, one root is infinite, and there are two actual points of intersection. For x large, the curve tends to assume the form $y=-1$, which explains the infinite root. When $c=-1$, equation (5) becomes

$$0 \, . \, x^3+0 \, . \, x^2+4x+3=0.$$

That is, the point at infinity on the curve in the direction of the x-axis, is a point of simple concavity at infinity, and the equation to the corresponding tangent or asymptote is $y=-1$.

11. In the case of a curve passing through the origin of coordinates, the appearance of the curve at the origin is obtained by making successive approximations of the form $y=ax$, $y=ax+bx^2$, etc., and thus the nature of a simple concavity, a point of inflexion, a cusp, a node, a conjugate point at any actual point is explained. When x is large, the corresponding successive approximations to the equation to the curve will be of the form

$$y=ax+b, \quad y=ax+b+\frac{c}{x}, \quad y=ax+b+\frac{c}{x^2}, \text{ etc.,}$$

and from such equations are investigated the appearance of a curve at infinity in any chosen direction.

12. To investigate the appearance of a curve at infinity when there is a simple concavity at a point at infinity in the plane of the curve, consider the curve given by the equation $y = ax + b + \frac{c}{x}$ (fig. 6, c +ve). The curve has a simple concavity at the point at infinity on the line $y = ax + b$, and also a simple concavity at the point at infinity on the line $x = 0$. Hence corresponding to a simple concavity at infinity, the curve approaches its asymptote from *opposite sides* of the line at the two extremities.

13. The case of a *point of inflexion* at infinity arises in the curve given by the equation $y = ax + b + \frac{c}{x^2}$, (fig. 7, c +ve). The line $y = ax + b$ meets the curve at points whose abscissae are given by the equation $c = 0$, where a cubic is in question; that is, the line meets the curve at *three* coincident points at infinity. And any other straight line through the point of contact of curve and line, being a parallel straight line, has an equation of the form $y = ax + k(k \neq b)$ and therefore meets the curve in two finite points (the abscissae of which are determined from the equation $(k - b)x^2 = c$) and in one point at infinity. The point at infinity in the direction $y = ax + b$ has therefore precisely the characteristics of a point of inflection at an actual point on a curve of the third degree, say the origin on the curve $y = x^3$, and is a point of inflection at infinity. In such a case we see from the graph that the curve appears at *both ends* of the asymptote $y = ax + b$, but on the *same side of* the *line*.

14. The same curve has a *cusp at infinity* in the direction given by the equation $x = 0$. The line $x = 0$ meets the curve in *three* coincident points at infinity. Any straight line through the point of contact except $x = 0$ is given by the equation $x = k$, and the ordinates of its intersection with the curve are given by the equation $y = ak + b + \frac{1}{k^2}$, a linear equation when a cubic is in question. Hence *two* of the points of intersection are coincident with the point at infinity on $x = k$ or $x = 0$, and the third is the finite point $(k, ak + b + 1/k^2)$. Hence this point at infinity is a double point. It is a cusp since the tangents at the point are real and coincident. Any line through the point may be explained geometrically to meet

each of the two branches, the particular line $x = 0$ touching one branch and meeting the other. From the graph we see that at a cusp at infinity, the curve appears only at one end of the asymptote, and on both sides at that end.

15. There is a *node at infinity* on the curve given by the equation $y = ax + b + c/(x^2 - 1)$, fig. 8, c + ve. As in § 13, there is a point of inflection at infinity on $y = ax + b$, and the curve appears on the same side of the line at both ends. But $x + 1 = 0$ also meets the curve in *three* coincident points at infinity, and so does $x - 1 = 0$. Any other line through this point at infinity meets the curve there in two coincident points (one on each branch), and in one finite point. The point is a double point, and $x \pm 1 = 0$ are real tangents at the point, which is therefore a node. The curve appears at both ends and on opposite sides of each asymptote.

16. There is a *conjugate point* at infinity on the curve whose equation is $y = ax + b + c/(x^2 + 1)$, fig. 9, c + ve. $x \pm i = 0$ are the imaginary asymptotes at the point.

17. To illustrate the method of finding the asymptotes of a given curve, consider that given by the equation $xy(x + y) + x^2 + y^2 = 0$. The lines $x = 0$, $y = 0$, $x + y = 0$ meet the curve in points whose abscissae or ordinates are given by the equations $y^2 = 0$, $x^2 = 0$, $2x^2 = 0$ respectively. These are quadratic equations where cubic equations are in question; therefore each of these lines meets the curve in *one* point at infinity. Hence any straight line parallel to one of these meets the curve in a point at infinity, since parallel lines have a common point at infinity. To find asymptotes, it remains to select (if possible) the particular parallels which meet the curve in two or more points at infinity.

(1) Consider the points of intersection of $x = k$ $(k \neq 0)$ and $xy(x + y) + x^2 + y^2 = 0$. The ordinates of these points are given by $y^2(1 + k) + k^2y + k^2 = 0$. Choosing $1 + k = 0$, we see that the ordinates of the points of intersection of $x + 1 = 0$, and the curve are given by the linear equation $y + 1 = 0$, where a cubic is in question. Therefore $x + 1 = 0$ meets the curve in two coincident points at infinity and at the finite point $(-1, -1)$. That is, $x + 1 = 0$ is an asymptote and its point of contact is a point of simple concavity at

infinity, and the curve will appear at both ends of $x+1=0$, and on opposite sides. By symmetry, similar remarks hold regarding the asymptote $y+1=0$.

(2) The points of intersection of $x+y=k$ $(k \neq 0)$ and the curve have their abscissae given by the equation $x^2(2-k)+kx(k-2)+k^2=0$. Hence the line $x+y=2$ meets the curve in points whose abscissae are given by the anomalous equation $4=0$, where a cubic is in question. The point at infinity on the line $x+y=2$ is a point of inflexion at infinity on the curve, and the curve will appear on the same side of the asymptote $x+y=2$, at its two ends.

We should therefore (§§ 12, 13) expect the approximate forms of the equation $xy(x+y)+x^2+y^2=0$ at infinity in the directions $x=0$, $y=0$, $x+y=0$ to be

$$x=-1+c/y, \quad y=-1+c/x, \quad y=-x+2+c/x^2 \text{ respectively.}$$

The first approximation to the equation to the curve at infinity in the direction $x=0$ is obtained by writing $x=\underset{x=0}{\mathrm{Lt}}\dfrac{x^2+y^2}{-y(x+y)}=-1$.

For the second approximation, write

$$x=\underset{x=-1}{\mathrm{Lt}}\frac{x^2+y^2}{-y(x+y)}=-\frac{1+y^2}{y(y-1)}=-1-\frac{1}{y},$$

for y large, by Descending Division.

Again write $x+y=\underset{y=-x}{\mathrm{Lt}}\dfrac{x^2+y^2}{-xy}=2$, which gives the first approximation at infinity in the direction $x+y=0$.

Next write $\quad x+y=\underset{y=-x+2}{\mathrm{Lt}}\dfrac{x^2+y^2}{-xy}=\dfrac{2x^2-4x+4}{x^2-2x}=2+\dfrac{4}{x^2}$,

for x large, by Descending Division.

Hence, in accordance with §§ 12, 13, the approximate forms of the equation at infinity are $x=-1-1/y$, $y=-1-1/x$, $x+y=2+4/x^2$; see fig. 10.

18. The parabola at infinity is of peculiar interest. Taking the equation in the form $y=ax^2$, we see that the point at infinity in the direction given by $x=0$ is a point on the curve. No straight line in this direction *touches* the curve and a straight line in any other

direction meets the curve in two finite points. Is there, then, no line which meets and touches the curve at infinity, that is, is there no asymptote? Let us find the limiting form of the equation to the tangent at the point (x', y') when $x' = k$, $y' = \infty$. Change the origin to the point (x', y'); the equation becomes $\eta + y' = a(\xi + x')^2$, or $\eta = 2ax'\xi + a\xi^2$. The equation to the tangent at the new origin is therefore $\eta = 2ax'\xi$ or $y - y' = 2ax'(x - x')$, which may be written in the form $y(1/y') + 1 = 2a(x'/y') \,.\, x$. In the limiting case we get $0 \,.\, y + 1 = 0 \,.\, x$, which is the line at infinity. Hence the line at infinity touches every parabola.

19. The line at infinity may touch other curves, so that these curves will touch certain parabolas at their points at infinity. $y(x-2)^2 = x^3(x-1)$, fig. 11, is an example of a curve which has parabolic contact at infinity. Using Descending Division, we may write the equation in the form $y = x^2 + 3x + 8 + \mathrm{R}$, where $\mathrm{R} = (20x - 32)/(x^2 - 4x + 4)$. Now $\operatorname*{Lt}_{x=\infty} \mathrm{R} \div 8 = 0$, therefore $y = x^2$ is a first approximation, $y = x^2 + 3x$ a second, and $y = x^2 + 3x + 8$ a third. All these approximate forms at infinity are parabolas with a point at infinity in the direction of the y-axis. The first meets the curve at one point at infinity, the second at two points at infinity, the third at three points at infinity; $y = x^2 + 3x + 8$ is a *parabolic asymptote*.

Third Meeting, 11th January 1907.

J. ARCHIBALD, Esq., M.A., President, in the Chair.

On the Cartesian Coordinates of Classes of Tortuous Curves.

By JOHN MILLER, M.A.

The following notation is here used for the quantities occurring in the discussion of tortuous curves.

Length $= s$; curvature $= \frac{1}{R}$; torsion $= \frac{1}{T}$; direction cosines of tangent, principal normal and binormal:—α, β, γ; l, m, n; λ, μ, ν.

Frenet's formulæ are therefore,

$$\frac{d\alpha}{ds} = \frac{l}{R}; \ \frac{dl}{ds} = -\frac{\alpha}{R} - \frac{\lambda}{T}; \ \frac{d\lambda}{ds} = \frac{l}{T}$$

with two corresponding sets.

To find the cartesians of a tortuous curve from the values of R and T as functions of s, Hoppe in *Crelle's Journal* (1862) reduced these equations to the discussion of a differential equation of the second order. Lie reduced them to a Riccati. The detailed process is given in Scheffers' *Einführung in die Theorie der Curven*, but only two cases are worked out: (i) the helix, (ii) the general helix on any cylinder that is the curves $\frac{R}{T} =$ a constant. These examples are trivial and need no elaborate theory. I shall return to the Riccati equation at the end of this paper.

Integral forms have been given for the cartesians of several classes of curves. Thus when R is a constant

$$x = R\int \alpha d\sigma, \quad y = R\int \beta d\sigma, \quad z = R\int \gamma d\sigma$$

where $d\sigma^2 = d\alpha^2 + d\beta^2 + d\gamma^2$, and $\alpha^2 + \beta^2 + \gamma^2 = 1$.

For T = a constant, Darboux gives the beautiful forms

$$x = \mathrm{T}\int\frac{ldk - kdl}{h^2+k^2+l^2}, \quad y = \mathrm{T}\int\frac{hdl - ldh}{h^2+k^2+l^2}, \quad z = \mathrm{T}\int\frac{kdh - hdk}{h^2+k^2+l},$$

where h, k, and l are arbitrary functions of a single variable.

For Bertrand's curves $\frac{a}{\mathrm{R}} + \frac{b}{\mathrm{T}} = c$ we have

$$x = \frac{a}{c}\int ud\sigma + \frac{b}{c}\int(wv' - vw')d\sigma,$$

$$y = \frac{a}{c}\int vd\sigma + \frac{b}{c}\int(uw' - wu')d\sigma,$$

$$z = \frac{a}{c}\int wd\sigma + \frac{b}{c}\int(vu' - uv')d\sigma,$$

where u, v and w are functions of σ such that $u^2+v^2+w^2=1$ and $u'^2+v'^2+w'^2=1$.

Finally Scheffers gives integral expressions of a very involved form for the curves $\frac{1}{\mathrm{R}^2}+\frac{1}{\mathrm{T}^2} = \text{constant}$.

It is seen that with the exception of the curves of constant torsion none of these can claim to be very explicit, and I can find no actual examples worked out except for T = constant. [See Darboux *Théorie des Surfaces*, Vol. IV., Appendix.] The first part of the present paper gives integral expressions of an explicit nature for these and other classes of curves by one uniform simple method which shows the reason of the occurrence of such integrals.

Let $x = \int\cos\theta ds, \quad y = \int\sin\theta\cos\phi ds, \quad z = \int\sin\theta\sin\phi ds.$

Then $\frac{1}{\mathrm{R}} = \sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}\frac{d\theta}{ds}$, the positive sign of the root being taken so that, as R is to be considered positive, θ and s increase together. If θ is a constant we shall have from

$$\frac{1}{\mathrm{R}} = \sqrt{\left\{\left(\frac{d\theta}{ds}\right)^2 + \left(\sin\theta\frac{d\phi}{ds}\right)^2\right\}}, \quad \frac{1}{\mathrm{R}} = \sin\theta\frac{d\phi}{ds}.$$

$$\frac{da}{ds} - \frac{l}{\mathrm{R}} = -\sin\theta\frac{d\theta}{ds}.$$

$$\therefore \quad l = -\frac{\sin\theta}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}.$$

$$\therefore \quad \lambda = \frac{\sin^2\theta\frac{d\phi}{d\theta}}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}.$$

$$\frac{dl}{ds} = -\frac{a}{R} - \frac{\lambda}{T}.$$

From this, $\dfrac{1}{T} = -\left\{\cos\theta\dfrac{d\phi}{d\theta} + \dfrac{d}{d\theta}\tan^{-1}\left(\sin\theta\dfrac{d\phi}{d\theta}\right)\right\}\dfrac{d\theta}{ds}.$

When R is a constant and θ is not a constant, the cartesians are

$$x = R\int\cos\theta\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta;$$

$$y = R\int\sin\theta\cos\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta;$$

$$z = R\int\sin\theta\sin\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta.$$

In these ϕ is an arbitrary function of θ.

I give some examples integrable in terms of ordinary functions.

1st Case. θ a constant. The spherical indicatrix is a circle and the curve is a helix.

$$x = \frac{R\phi}{2}\sin 2\theta, \quad y = R\sin^2\theta\sin\phi, \quad z = -R\sin^2\theta\cos\phi.$$

Let us transform the integrals by making $\tan\dfrac{\theta}{2} = v$ and $\tan\dfrac{\phi}{2} = u$ where u and v are arbitrary functions of a variable t.

$$a, \beta, \gamma \text{ are } \frac{1-v^2}{1+v^2}, \quad \frac{2v(1-u^2)}{(1+u^2)(1+v^2)}, \quad \frac{4uv}{(1+u^2)(1+v^2)}.$$

Then
$$\frac{x}{R} = 2\int\frac{1-v^2}{(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt,$$

$$\frac{y}{R} = 4\int\frac{(1-u^2)v}{(1+u^2)(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt,$$

$$\frac{z}{R} = 8\int\frac{uv}{(1+u^2)(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt.$$

2nd Case. $v = 1 + u^2$ and $u^2 = t$. The spherical indicatrix is

$$\tan\frac{\theta}{2} = \sec^2\frac{\phi}{2}.$$

$$\frac{x}{R} = -2\int\frac{t^2+2t}{(t^2+2t+2)^2}\sqrt{\left(1+\frac{1}{t}\right)}dt,$$

$$\frac{y}{R} = 4\int\frac{1-t}{(t^2+2t+2t)^2}\sqrt{\left(1+\frac{1}{t}\right)}dt,$$

$$\frac{z}{R} = 8\int\frac{\sqrt{(1+t)}dt}{(t^2+2t+2)^2}.$$

3rd Case. $v = \dfrac{1+u^2}{u}$, that is $\tan\dfrac{\theta}{2} = 2\operatorname{cosec}\phi$.

$$\frac{x}{R} = -2\int\frac{(u^4+u^2+1)(1+u^2)udu}{(u^4+3u^2+1)^2},$$

$$\frac{y}{R} = 4\int\frac{(1-u^4)u^2du}{(u^4+3u^2+1)^2},$$

$$\frac{z}{R} = 8\int\frac{(1+u^2)u^3du}{(u^4+3u^2+1)^2}.$$

4th Case. $v = \dfrac{1+u^2}{u^2}$, that is $\tan\dfrac{\theta}{2} = \operatorname{cosec}^2\dfrac{\phi}{2}$.

$$\frac{x}{R} = -4\int\frac{u(2u^2+1)\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du,$$

$$\frac{y}{R} = 8\int\frac{(1-u^2)u^3\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du,$$

$$\frac{z}{R} = 16\int\frac{u^4\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du.$$

There is no need to give the results which involve logarithmic and inverse circular functions.

The locus of the centre of curvature (ξ, η, ζ) of a curve of constant curvature is a similar curve and, if T and T′ be the radii of torsion, $TT' = R^2$.

As an example, take the 2nd case.

$$ds = \frac{2R\sqrt{\left(1+\frac{1}{t}\right)}}{t^2+2t+2}dt; \quad a = -1+\frac{2}{t^2+2t+2}; \quad \frac{da}{ds} = -\frac{2\sqrt{t(1+t)}}{R(t^2+2t+2)} = \frac{l}{R}.$$

Similarly $m = \dfrac{t(t^2 - 2t - 4)}{2(t^2 + 2t + 2)\sqrt{t(1 + t)}}$, $n = \dfrac{2 - 2t - 3t^2}{(t^2 + 2t + 2)\sqrt{(1 + t)}}$.

$$\xi = x + l\mathrm{R}, \quad \eta = y + m\mathrm{R}, \quad \zeta = z + n\mathrm{R}.$$

When T is a constant the cartesians are

$$x = -\mathrm{T}\int \cos^2\theta d\phi - \mathrm{T}\int \cos\theta d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right),$$

$$y = -\mathrm{T}\int \sin\theta\cos\theta\cos\phi d\phi - \mathrm{T}\int \sin\theta\cos\phi d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right),$$

$$z = -\mathrm{T}\int \sin\theta\cos\theta\sin\phi d\phi - \mathrm{T}\int \sin\theta\sin\phi d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

One integrable class occurs evidently when $\phi = n\theta + c$, n a positive integer. We give the results for $\phi = \theta$.

$$\frac{x}{\mathrm{T}} = \frac{\theta}{2} - \tfrac{1}{4}\sin 2\theta - \frac{1}{\sqrt{2}}\cos^{-1}\frac{3\cos 2\theta - 1}{3 - \cos 2\theta},$$

$$\frac{y}{\mathrm{T}} = \tfrac{1}{3}\cos^3\theta - \cos\theta + \frac{1}{\sqrt{2}}\log\frac{\sqrt{2} + \cos\theta}{\sqrt{2} - \cos\theta},$$

$$\frac{z}{\mathrm{T}} = -\tfrac{1}{3}\sin^3\theta - \sin\theta + \tan^{-1}\sin\theta.$$

It may be easily proved that $\mu = \dfrac{\sin\phi + \cos\theta\cos\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{\sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$

and that $\nu = \dfrac{\cos\phi - \sin\phi\cos\theta\left(\sin\theta\frac{d\phi}{d\theta}\right)}{\sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$.

By making

$$\frac{\sin^2\theta\frac{d\phi}{d\theta}}{h\sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}} = \frac{\sin\psi + \cos\theta\cos\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{k\sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$$

$$= \frac{\cos\phi - \cos\theta\sin\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{l\sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}} = \frac{1}{\sqrt{(h^2 + k^2 + l^2)}}$$

we get Darboux's formulae.

For Bertrand's curves we have

$$ds = \frac{a}{c}\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta + \frac{b}{c}\cos\theta d\phi + \frac{b}{c}d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

$$x = \frac{a}{c}\int\cos\theta\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta + \frac{b}{c}\int\left\{\cos^2\theta\frac{d\phi}{d\theta}+\cos\theta\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta,$$

$$y = \frac{a}{c}\int\sin\theta\cos\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta + \frac{b}{c}\int\sin\theta\cos\phi\left\{\cos\theta\frac{d\phi}{d\theta}+\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta,$$

$$z = \frac{a}{c}\int\sin\theta\sin\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta + \frac{b}{c}\int\sin\theta\sin\phi\left\{\cos\theta\frac{d\phi}{d\theta}+\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta.$$

For shortness let

$$\mathrm{P} = \sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}} \text{ and } \mathrm{Q} = -\cos\theta\frac{d\phi}{d\theta} - \frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

For the curves whose radius of screw (Frost) is constant, that is the curves $\frac{1}{\mathrm{R}^2}+\frac{1}{\mathrm{T}^2} = \text{constant} = \frac{1}{c^2}$,

$$ds = c\sqrt{(\mathrm{P}^2+\mathrm{Q}^2)}d\theta$$

$$= c\sqrt{\left[1+\left(\frac{d\phi}{d\theta}\right)^2 + 2\cos\theta\frac{d\phi}{d\theta}\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \left\{\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}^2\right]}d\theta.$$

The curves whose principal normals are the binormals of a second curve,

$$\frac{1}{\mathrm{R}^2}+\frac{1}{\mathrm{T}^2}=\frac{1}{a\mathrm{R}}$$

give

$$ds = a\left(\mathrm{P}+\frac{\mathrm{Q}^2}{\mathrm{P}}\right)d\theta.$$

The curves $\frac{a}{\mathrm{RT}}+\frac{b}{\mathrm{T}^2}+\frac{c}{\mathrm{R}^2}+\frac{d}{\mathrm{R}}=0$, the axis of whose osculating helix with the same torsion describes, with reference to the tangent, principal normal and binormal as axes, a Plücker's conoid [Demoulin, *Paris Soc. Math. Bull.* 21 (1893)] give

$$ds = -\frac{1}{d}\left\{a\mathrm{Q}+\frac{b\mathrm{Q}^2}{\mathrm{P}}+c\mathrm{P}\right\}d\theta.$$

The curves $\frac{a}{\mathrm{RT}}+\frac{b}{\mathrm{T}^2}+\frac{c}{\mathrm{R}^2}+\frac{d}{\mathrm{R}}+\frac{e}{\mathrm{T}}=0$ in which a straight line fixed relatively to the tangent, principal normal and binormal generates a developable surface [E. Cesàro, *Natürliche Geometrie*] give

$$ds = -\{a\mathrm{PQ}+b\mathrm{Q}^2+c\mathrm{P}^2\}d\theta/(d\mathrm{P}+e\mathrm{Q}).$$

Hence expressions as integrals, although very cumbrous, can be given for the cartesians.

Let us now consider the curves $\frac{R}{T}=f(s)$ given by Enneper (*Mathematische Annalen*, 1882) and Pirondini (*Crelle's Journal* 1892) as geodetics on developable surfaces.

If $\frac{d\theta}{\sin\theta}=du$, that is $\tan\frac{\theta}{2}=e^{u}$,

then
$$\frac{1}{R}=\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}\frac{1}{\cosh u}\frac{du}{ds},$$

and
$$-\frac{1}{T}=\left\{-\tanh u\frac{d\phi}{du}+\frac{d}{du}\tan^{-1}\left(\frac{d\phi}{du}\right)\right\}\frac{du}{ds}.$$

$$\therefore\ -\frac{R}{T}=-\frac{\sinh u\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}+\cosh u\frac{d}{du}\left[\frac{\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}\right].$$

$$\therefore\quad \frac{\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}=\left\{\text{const.}-\int\frac{R\,du}{T\cosh^2 u}\right\}\cosh u.$$

Hence if $\cosh u$ be made an arbitrary function of s, $\frac{d\phi}{du}$ and therefore ϕ are known in terms of s. The whole is now reduced to a question of quadratures.

Let $\sin\theta\frac{d\phi}{d\theta}=\tan\zeta$ where ζ is an arbitrary function of θ.

Then $\frac{1}{R}=\sec\zeta\frac{d\theta}{ds}$ and $-\frac{1}{T}=\left\{\cot\theta\tan\zeta+\frac{d\zeta}{d\theta}\right\}\frac{d\theta}{ds}$.

$$\therefore\quad \phi=\int\frac{\tan\zeta}{\sin\theta}d\theta,$$

$$\int\frac{ds}{R}=\int\frac{d\theta}{\cos\zeta},$$

$$-\int\frac{ds}{T}=\zeta+\int\frac{\tan\zeta}{\tan\theta}d\theta.$$

The elimination of ζ or θ gives an involved differential equation of the second order for θ or ζ. We may, however, get some solutions

when R or T is given as a function of s by giving a value to ζ. The corresponding value of T or R can then be got. Thus if $\zeta=\theta$, $\phi=\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ and $\int\frac{ds}{\mathrm{R}}=\log c\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$. Hence, if $\mathrm{R}=s$ so that $s=c\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ and $\phi=\log\frac{s}{c}$, then $\mathrm{T}=-\frac{c^2+s^2}{4c}$.

Again if $\phi=\theta$ and $\mathrm{R}=c\sqrt{\left(\frac{c^2-s^2}{c^2+s^2}\right)}$ we have $\sin\theta=\frac{s}{c}$ and $\mathrm{T}=-\frac{c(c^2+s^2)}{2c^2+s^2}$.

The coordinates are then

$$x=c\int\cos^2\theta d\theta = \frac{c}{2}(\theta+\sin\theta\cos\theta),$$

$$y=c\int\sin\theta\cos^2\theta d\theta=-\frac{c}{3}\cos^3\theta,$$

$$z=c\int\sin^2\theta\cos\theta d\theta=\frac{c}{3}\sin^3\theta.$$

We wish here to give a note on the Riccati equation given by Lie, although the work is not directly connected with the preceding method. Before giving his substitution we shall slightly change the form of Frenet's formulae by writing $\mathrm{R}=\frac{ds}{d\theta}$ or alternatively $\mathrm{T}=\frac{ds}{d\omega}$ where $d\theta$ and $d\omega$ are the angles of contingence and torsion.

Thus $$\frac{da}{d\theta}=l,\quad \frac{dl}{d\theta}=-a-\frac{\mathrm{R}}{\mathrm{T}}\lambda,\quad \frac{d\lambda}{d\theta}=\frac{\mathrm{R}}{\mathrm{T}}l, \text{ etc.,}$$

or $$\frac{da}{d\omega}=\frac{\mathrm{T}}{\mathrm{R}}l,\quad \frac{dl}{d\omega}=-\lambda-\frac{\mathrm{T}}{\mathrm{R}}a,\quad \frac{d\lambda}{d\omega}=l, \text{ etc.}$$

Then, as pointed out by Hoppe, from any expressions involving a, λ, R, T and θ we get corresponding ones with ω by interchanging a and λ, R and T, θ and ω,

Since $$a^2+l^2+\lambda^2=1$$

we may put $$\frac{a+il}{1-\lambda}=\xi.$$

Then $$\frac{d\xi}{d\omega}=\frac{i}{2}(1-\xi^2)-i\frac{\mathrm{T}}{\mathrm{R}}\xi.$$

If $\phi + i\psi$ is a particular solution, ϕ and ψ being real functions of ω,

$$\frac{d\psi}{d\omega} = \tfrac{1}{2}(1 - \phi^2 + \psi^2) - \frac{T}{R}\phi,$$

and
$$\frac{d\phi}{d\omega} = \phi\psi + \frac{T}{R}\psi.$$

By elimination of $\dfrac{T}{R}$

$$\phi\frac{d\phi}{d\omega} + \psi\frac{d\psi}{d\omega} = \tfrac{1}{2}\psi(1 + \phi^2 + \psi^2).$$

$\therefore\ 1 + \phi^2 + \psi^2 = ce^{\int \psi d\omega}$, c a positive constant.

If ϕ or ψ be taken arbitrarily, a particular solution is determined and the general solution can be found corresponding to the value $\dfrac{1}{\psi}\dfrac{d\phi}{d\omega} - \phi$ of $\dfrac{T}{R}$. We refer to Scheffers' *Einführung in die Theorie der Curven* for the deduction of coordinates from the general integral. To take a simple case, let $\phi = \cosh\theta$, $\psi = -\sin\theta$ and $c = 2$. By changing the variable from ω to θ, the equation is

$$\frac{d\xi}{d\theta} = \frac{iR}{2T}(1 - \xi^2) - i\xi.$$

From $\quad 1 + \phi^2 + \psi^2 = 2e^{\int \psi d\omega}$

we have
$$\frac{d\omega}{d\theta} = -\frac{2}{\cosh\theta}.$$

$\therefore$
$$\frac{T}{R} = -\tfrac{1}{2}\cosh\theta.$$

The equation now takes the form

$$\frac{d\xi}{d\theta} + \frac{i(1 - \xi^2)}{\cosh\theta} + i\xi = 0$$

of which a particular solution is $\cosh\theta - i\sinh\theta$.

The general solution is therefore

$$\frac{d\{\cosh\theta - i\sin\theta\} + e^{i\theta}}{d - e^{i\theta}\{\cosh\theta + i\sinh\theta\}}$$ where d is a constant.

The coordinates are

$$x = -\int \tanh\theta \sin\theta \mathrm{R} d\theta,$$

$$y = -\int \tanh\theta \cos\theta \mathrm{R} d\theta,$$

$$z = -\int \operatorname{sech}\theta \mathrm{R} d\theta,$$

where R is an arbitrary function of θ.

When a case is worked out with ω as the variable, T will occur in the final integrals. Hence if R or T can be expressed as a function of $\frac{\mathrm{T}}{\mathrm{R}}$, as in the classes of curves given earlier, we shall have from any given solution of the equation an example of each of such classes.

On the Arithmetic and Geometric Means Inequality.

By V. Ramaswami Aiyar, M.A.

Let a and b be positive and unequal; and let x be their arithmetic mean. Then we have the following equality and inequality

$$a - x = x - b$$

$$\frac{a}{x} < \frac{x}{b}.$$

The inequality is tantamount to the fact that the G.M. of a and b is less than the A.M.

Instead of the single arithmetic mean, let us consider $p+q-1$ arithmetic means inserted between a and b. Let these be $x_1, x_2, \ldots x_{p+q-1}$. Then we have by the above

$$a - x_1 = x_1 - x_2 = \ldots = x_{p+q-1} - b \quad - \quad - \quad - \quad (1)$$

and

$$\frac{a}{x_1} < \frac{x_1}{x_2} < \ldots\ldots\ldots < \frac{x_{p+q-1}}{b}. \quad - \quad - \quad - \quad - \quad (2)$$

(1) and (2) each contain $p+q$ members. The A.M. of the first q members in (1) is equal to the A.M. of the remaining p members, since all the members are equal. Hence

$$\frac{a - x_q}{q} = \frac{x_q - b}{p}.$$

The G.M. of the first q members in (2) is less than the G.M. of the remaining p members, since each of the former is less than each of the latter. Hence

$$\left(\frac{a}{x_q}\right)^{\frac{1}{q}} < \left(\frac{x_q}{b}\right)^{\frac{1}{p}}. \quad - \quad - \quad - \quad - \quad - \quad (4)$$

From (3) we get $x_q = (pa+qb)/(p+q)$ and from (4) $x_q > (a^p b^q)^{1/(p+q)}$. Hence we have

$$\frac{pa+qb}{p+q} > (a^p b^q)^{\frac{1}{p+q}}. \quad - \quad - \quad - \quad - \quad - \quad (5)$$

Having proved this, we could now generalise and show that if $p, q, \dots t$ be any n positive rational numbers and $a, b, \dots k$ any n positive quantities, not all equal, then

$$\frac{pa + qb + \dots + tk}{p + q + \dots + t} > (a^p b^q \dots k^t)^{\frac{1}{p+q+\dots+t}}.$$

If y be the geometric mean of a and b we have

$$\frac{a}{y} = \frac{y}{b}, \quad a - y > y - b.$$

Starting from this equality and inequality and proceeding in like manner, we get result (5). In this way we have another demonstration of the arithmetic and geometric means inequality.

The trick employed here is of a more general application. The following inequalities, for example, can be deduced in this way from the particular cases of them when $n = 2$, in which case they are easily shown to be true.

(i) If A, B, ... K be n positive acute angles, not all equal, then

$$(1) \quad \frac{\sin A + \sin B + \dots + \sin K}{n} < \sin \frac{A + B + \dots + K}{n}$$

$$(2) \quad \frac{\cos A + \cos B + \dots + \cos K}{n} < \cos \frac{A + B + \dots + K}{n}$$

$$(3) \quad \frac{\tan A + \tan B + \dots + \tan K}{n} > \tan \frac{A + B + \dots + K}{n}.$$

(ii) If A, B, ... K be n positive angles, each less than half a right angle, and not all equal, then

$$\tan \frac{A + B + \dots + K}{n} > \sqrt[n]{\tan A . \tan B \dots \tan K}.$$

Coaxial Circles and Conics.

By W. FINLAYSON.

The following notes are intended to introduce a simple method of treating elementary geometrical conics, and at the same time to supply a missing link in the chain of continuity between Euclidean geometry and the modern methods of treating the conics, which at present are treated more as different subjects than as a continuous whole.

FIGURE 12.

§ 1. Let Xx be the line of centres of the coaxial system, whose common points are S and S_1, and the radical axis of the system orthogonal to the first; then S and S_1 are common inverse points to the second system. Let F be the centre of any circle of the second system, and x the centre of any circle of the first. The tangents to x, or any circle of the x-system, from F are then of constant length, being always equal to the radius, R, of F. Let Ff_1 and Ff_2 be the two tangents from F to x, QSP and $Q'S_1P'$ the tangents to x at S and S_1, cutting the tangents to x from F in P, Q, P′, Q′. Then, since Pf_1 and PS are tangents to x from P, $Pf_1 = PS$. Therefore $PF - Pf_1 = PF - PS = R$ and $QF - QS = R$. Hence the locus of P or Q is a curve the difference of any point on which from two fixed points, F and S, is constant, that is an hyperbola.

Again, $P'f_1$ and $P'S_1$, being also tangents to x from P′, are equal. Therefore $P'F + P'S = P'F + P'f_1 = R$, and the locus in this case is an ellipse.

This gives a very simple method of tracing either curve. For x may be any point on Xx and, x being the centre of the circle to which SP or S_1P' is a tangent, xSP, or xS_1P', is always a right angle. Take therefore any point x_a; join x_aS; draw at S a line at right angles to x_aS and, with x_aS as radius, mark f_a on the circumference of F; draw Ff_a cutting SP_a in P_a, and so on.

FIGURE 12.

§ 2. Let PM be the perpendicular from P on the radical axis, Xx, of the F-system, and PK a tangent to F from P; then, by a well-known property of coaxial circles, we have

$$2SF \,.\, PM = PK^2 - PS^2 \quad - \quad - \quad - \quad - \quad (1)$$

S being a point circle of the F-system.

Now $\qquad PK^2 = (Pf_1 + R)^2 - R^2$

and $\qquad Pf_1 \;= PS.$

Putting these values in (1) we get

$$2SF \,.\, PM = R^2 + 2PS \,.\, R + PS^2 - R^2 - PS^2;$$

therefore $\qquad SF \,.\, PM = PS \,.\, R$ or $PM : PS = R : SF.$

Hence $CA : CS = PM : PS$ and, R and SF being fixed quantities, $PM : PS$ is a constant ratio. Thus we see that the radical axis is also the directrix.

For the ellipse, the proof is slightly different. P′ being inside the circle F, take $P'K'$ (not shown in figure) at right angles to $P'f_1$;

then $\qquad 2S_1F \,.\, P'M' = P'K'^2 + P'S_1^2. \quad - \quad - \quad - \quad (2)$

Now $P'K'^2 = R^2 - (R - P'f_1)^2$ and $P'f_1 = P'S_1$.

Substituting these values in (2), we have

$$2S_1F \,.\, P'M' = R^2 - R + 2P'S_1 \,.\, R - P'S_1^2 + P'S_1^2$$

or, as before, $S_1F \,.\, P'M' = P'S_1 \,.\, R$, and therefore $P'M' : P'S_1 = R : S_1F$ a constant ratio.*

When the two tangents of the F-system are taken parallel to SS_1, the points of contact f_1 and f_2 will be on xX. But PM will then be equal to $Pf_1 = PS$ and therefore $PM : PS = 1$; so that in all cases the radical axis is the directrix. In the last case, that of the parabola, the orthogonal, or director circle, coincides with the radical axis, F being at infinity.

FIGURE 12.

§ 3. The line xP, or xP', from the centre x to the point of contact, is the tangent to the curve at P, or P′. Since it bisects S_1f_1 at

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* See the appended proof that $2S_1F \,.\, P'M' = P'K^2 + P'S_1^2$.

right angles, it is the locus of points from which equal lines can be drawn to S_1 and f_1. Taking, therefore, any point P_1 on it, we have $P_1S_1 = P_1f_1$. But $P_1f_1 + P_1F$ is greater than R, and therefore greater than $P'S_1 + P'F$. Hence no other point on the line xP', except P', is on the curve.

A similar proof applies to the hyperbola and the parabola.

FIGURE 12.

§ 4. The proofs of the following standard propositions become extremely simple by this method.

(i) The tangent to a conic bisects the angle between the focal distances of the point of contact. For FP and SP are tangents to x and therefore xP bisects the angle between them.

(ii) Tangents at the ends of a focal chord meet on the directrix. For they both pass through the centre of x which is on the directrix.

(iii) The intercept on the tangent between P and the directrix subtends a right angle at the focus. For SP is a tangent and xS a radius at S to x; hence xSP is a right angle.

(iv) Perpendiculars on the tangents to a conic from the focus cut the tangents on a fixed circle. For Sf_1 is always bisected at right angles by the tangent. The cut a is therefore always the mid-point of the line from S to the circumference of F, and hence the locus of a is a circle whose radius is $\frac{1}{2}$R and whose centre C is the mid-point of SF. (See Mackay's *Euclid*, Appendix I., Prop. 6.)

(v) The normal at P cuts the axis at G so that

$$SG : SP = SP : PM.$$

For, since PG is parallel to Sf_1,

$$SG : Pf = FS : R = SP : PM \text{ and } Pf_1 = SP.$$

Hence

$$SG : SP = SP : PM.$$

The proofs of many other standard theorems can be much simplified in the same way.

FIGURE 12.

§ 5. Since P is clearly the pole of Sf_1 to x, the conic is the locus of the poles of the varying chord Sf of the varying circle x.

FIGURE 13.

§6. In figure 12 it is evident that, S_1 being inside F, S_1P' will always cut Ff internally to F; hence P' will always be internal to F, and the curve is therefore closed and finite, lying wholly inside the circle F. But when S is external to F, the intersection of Ff, and SP will move away from f_1 as the angle FPS becomes less, *i.e.*, as x increases in radius; and when Sf_1 becomes the tangent to F from S, it is evident that FP and SP, being at opposite ends of the diameter Sf_1 of the circle x, will be parallel, and therefore xP will also be parallel to them. The point P will then have no existence or be at infinity, and, however far xP be produced, it can never meet the curve; for FP and SP can never meet. It is evident from fig. 12, however, that P continually approaches the asymptote; for the point a continually approaches x until it coincides with it, when Sf_1 becomes the tangent to F. When Sf_1 becomes the tangent to F, it is bisected by Xx, and xP_∞, being parallel to Ff_1, passes through C, the mid-point SF; and $Cx = \frac{1}{2}R$. Hence the asymptote, directrix, and auxiliary circle all pass through x. Since a second tangent can be drawn to F from S, another asymptote exists, equally inclined to SS_1 but in an opposite sense of direction. Thus each hyperbola has two asymptotes symmetrically situated with regard to the axis and passing through the centre C.

FIGURE 14.

§7. We might infer from the symmetrical position of the asymptotes and focal radii in the last note that the curve would have a corresponding branch on the opposite side of C. But we have only to continue the process of construction of figure 12 to arrive at the locus of the second branch. For when x is taken at a greater distance from X than the cut of the tangent to F from S, then P_1 will fall on the tangent from S on the opposite side of S from P so that $\qquad SP_1 = P_1f_1$;

whence again $\qquad P_1S - P_1F = Ff_1 = R$

and $\qquad SA' - A'F = Ff = R$.

From this we derive $SA' = SA$ and $CA' = CA$;

showing that F has the same relation to the second branch that S has to the first, and that the asymptotes to the first branch are also asymptotes to the second.

§ 8. There being an infinite series of circles orthogonal to the x-system on each side of the radical axis, there will therefore be a double infinite series of conics, related in pairs, corresponding to the circles similar to the two discussed; an ellipse in every circle and a double-branched hyperbola to every two equal circles which have their centres equidistant from X.

FIGURE 12.

§ 9. The radius of the circle F is cut harmonically by the ellipse and hyperbola, that is, $FP'F_1P$ and $FQ'f_2Q$ are harmonic ranges. For the two tangents at S and S_1 meet on the directrix at Z, and F lies on the polar of Z to x; therefore Z lies on the polar of F to x; and hence FS_1OS is harmonic, ZO, the polar of F, passes through f_1 and f_2, and, the pencil $Z(FS_1OS)$ being harmonic, the ranges $FP'f_1P$ and $FQ'f_2Q$ are harmonic. Similarly $F(f_1Of_2Z)$ is harmonic, and therefore $ZQ'S_1P'$ and ZQSP are harmonic. Hence any line through F cutting the ellipse, hyperbola, and circle, is divided harmonically, and the tangents at S and S_1 are divided harmonically by the curve, the focus, and the directrix.

FIGURE 13.

§ 10. The range $FP'f_1P_\infty$ being harmonic and P_∞ being at infinity, $FP' = f_1P'$ and similarly $ZQ = SQ$. As commonly put in the theory of harmonics, this results from the equation

$$FP : f_1P' = FP_\infty : f_1P_\infty .$$

This is, however, not an equality; for the difference of FP_∞ and f_1P_∞ is clearly $Ff_1 = R$. But in view of the fact that the harmonic conjugate to a point in a segment changes its direction when the point passes from one side of the centre of the segment to the other, if we take $P_{1\infty}$ in an opposite sense from P_∞, we have an absolutely true equation $FP' : fP' = FP_\infty : f_1P_{1\infty}$ and $FP_\infty = f_1P_{1\infty}$. The two equalities $FP' = f_1P'$ and $ZQ = SQ$ can, however, be easily proved otherwise. For $FP' = f_1P'$, we have, when Sf_1 is the tangent from the inverse of the focus to the circle F, S_1f_1 is at right angles to the axis. Therefore xP', which is at right angles to S_1f_1 through its mid-point a, bisects Ff_1. For $ZQ = SQ$, we have that x is the mid-point of Sf_1 and f_1Z is parallel to xQ, both being at right angles to Sf_2, and f_1Z passes through f_2; therefore $ZQ = QS$.

FIGURE 13.

§ 12. The following simple relations may be noted.

S_1f_1 is equal to the minor axis of the ellipse, and Ff_1, or R, is equal to the major axis.

S_1F, the distance between the foci, forms with S_1f_1 and Ff_1 a right-angled triangle; whence $R^2 - BB_1^2 = S_1F^2$. In the hyperbola $R^2 + BB^2 = SF^2$. Halving these lines, we have the usual forms

$$CA^2 - CB^2 = CS_1^2, \quad CA^2 + CB^2 = CS^2.$$

Again $FS_1 . FS = R^2$. Taking the mid-points of FS_1 or FS and SS_1, we have $CS . CX = CA^2$. Therefore Xx is the polar of S or S_1 to C the auxiliary circle.

§ 13. It is easy to extend these notes to show how the second focus and directrix can be found and a second coaxial system of circles along with them. The only other point, however, which need be noted just now is that there is one, and only one, circle which belongs to both directrices. Its centre is C, and the radius evidently is $\sqrt{CA^2 \pm CB^2}$ or $CR^2 = CA^2 \pm CB^2$. It is this circle which is sometimes called the director circle or orthocycle, and which is the locus of intersection of tangents at right angles to each other. It is the doubly orthogonal circle which has the directrices for radical axis and the foci and their inverses in common with the director circles F and S_1 or S and the systems to which they belong.

FIGURE 16.

§ 14. By varying the position of F, we obtain a clear view of how the curves of the three classes—ellipse, parabola, and hyperbola—are related to each other, taking two circles of the x-system and drawing the tangents to them at S, namely, SZ and SZ_1. (In the figure SZ_1 is taken at right angles to the axis, and therefore determines the latus rectum in each case.)

The tangents to X and x from F determine points P, P,′ P_1, P_1' on an ellipse greater or less as F is taken further from or nearer to S. At S we have a point circle as the limit. The ellipse increases as F is taken further and further from S, until we reach the parallel position at infinity, when the curve developes into a parabola and

F changes sides and appears as $-$F. The tangents from $-$F then determine points Q, Q$'$, Q_1, Q_1' on a hyperbola which, when $-$F is taken at S_1, has the directrix Xx for its limit; for then the tangents at S intersect the tangents at S_1 on the directrix.

§ 15. The property of the parabola that SA = AX has the corresponding property in the other conics that SA = Af, as Xx is in the case of the parabola the limit circle of the orthogonal circles of the F-system.

FIGURE 17.

§ 16. Being given the foci F and S, and R, either as the sum of the focal distances, or from R : SF = PM : PS, we can find the orthogonal system with its radical axis related to the conic as in § 1.

For let F be a circle of given radius and S_1 any point on Ff. Through the ends of any diameter of F, such as f_3f_4, draw chords through S_1; these cut F again in f_1 and f_2. Produce f_4f_1 to meet f_3f_2 in S_2. Then, because of the right angles at f_1 and f_2, $S_1f_1S_2f_2$ are cyclic and Fx bisects the angles f_1Ff_2 and $f_1S_2f_2$. But half these angles equals a right angle; for they are equal to $f_2f_4S_2 + f_2Sf_4$, and therefore Ff_1x and Ff_2x are right angles. Hence x is orthogonal to F, and xX, which is at right angles to FS_1, is the directrix. We might have taken S as any external point, such as the inverse of S_1, in which case the diameter would be that indicated by the line f_5f_6, making equal angles with the directrix but in an opposite sense to f_3f_4.

FIGURE 17.

§ 17. The locus of S_2, the intersection of f_4f_1 and f_3f_2, is the polar of S_1. For S_1SS_2 is a right angle and S is the inverse of S_1.

FIGURE 17.

§ 18. $P'S_1Q'$ is parallel to the diameter through the ends of which S_1f_1 and S_1f_2 pass.

For $$\angle P'S_1f_1 = P'f_1f_3 = f_1f_3F$$

and therefore the focal chord $P'S_1Q'$ and the diameter f_3f_4 are parallel.

FIGURE 17.

§ 19. If C be the auxiliary circle and xP′, xQ′ the tangents to a focal chord P′S$_1$Q′, the tangents cut C again at the ends of a parallel diameter. For S$a = af_1$, and aP′ is parallel to f_1f_3; therefore Sf_4 is bisected at c by the tangent and, since cac_1 and $ca_1'c_1$ are at right angles in the same semicircle, the tangent xa_1Q$_1$C$_1$ passes through the end of the diameter cCc_1 and, since aP′ is parallel to f_1f_4, cCc_1 is parallel to f_2f_4. This gives a very ready method of determining the point of contact of a given tangent. For let ac be the tangent; then SP parallel to Cc gives the required point, and a and c can by this always be found if R is given.

FIGURE 17.

§ 20. (a) xF bisects $\angle$ P′FQ′; for it bisects f_1Ff_2 and we know P′S$_1x$ and Q′S$_1x$ to be both right angles, hence xS bisects P′S$_1$Q′.

(b) Cx bisects P′Q′; for C is the mid-point of the base cc_1 of xcc_1 and P′Q′ is parallel to cc_1.

From (a) and (b), by aid of a generalisation of some elementary geometry of the circle we can get the general case of tangents from a point bisecting the focal angle of points of contact, and the line xC (or xy if y be the point on second direction) bisecting all chords of a conic parallel to PSQ or cc_1.

FIGURE 17.

§ 21. Since the tangent P′H is parallel to f_1f_4, and the normal P′J is parallel to f_1f_3, we have FH and FJ each equal FP′. Hence the tangent and normal at any point P′ can be determined very easily. For draw JH through F parallel to SP′; then a circle, with FP′ as radius, cuts this line in J and H which, on being joined to P′, give the required lines. This holds for conics generally, as nearly the whole of these notes do.

FIGURE 17.

§ 22. The following points are too obvious to require proof.

(1) The triangle xcc_1 has its sides half the parallel sides of S$_2f_4f_3$.

(2) The focus is the orthocentre or polar centre of both triangles.

(3) The cut of the polar of the inverse of the focus to F and C determines the polar radius Sk and Sk_1, that is, a perpendicular to FS_1 at S_1 cuts F and C in k and k_1 and Sk and Sk_1; this, in the case of $S_1 f_2 f_3$, is equal to the minor axis BB_1 and, in the case of xcc_1, half the minor axis or CB.

(4) xcc_1 is a self-conjugate triangle. For cc_1 is the polar of x, c_1x is the polar of c and c is the polar of c_1. So that any triangle is self conjugate.

FIGURE 17.

§ 23. Since $CS = \frac{1}{2}FS$, $SX = \frac{1}{2}SS_1$, $CA = \frac{1}{2}R$, $FS . FS_1 = R^2$, we have $CS . CX = CA^2$ and therefore xX is the polar of S to C.

If the tangent at P′ were produced to cut the axis in T, and a perpendicular from P′ cuts the axis in N, then, since P′, N, S_1, a are cyclic, cC$S_1 a$ are also cyclic and, C being the centre and Ca a chord cutting the diameter CN in T, we have

$$CN . CT = CA^2 = CSCX.$$

Hence TaPc is harmonic and therefore, if NP cut C in t, tT is the tangent to C from T; or, in the case of the hyperbola, if the perpendicular from T cut C in t, tN is the tangent from N to C.

FIGURE 17.

§ 24. If through c and c_1 lines cY and c_1Y be drawn parallel to xc_1 and xc, then yY, perpendicular to SF, is the second directrix. For YF = xS and so on, the position of y being identical with that of x to every line in the figure, if xcc_1 were revolved round C till c coincided with c_1.

FIGURE 17.

§ 25. Cc and Cx are conjugate diameters and are conjugate lines to the polar of circle xcc_1. For, by § 19 (a), Cx bisects P′Q′ which is parallel to cc_1.

Again, since x is the pole of cc_1 and C the pole of xX, the pole of Cx is the point R in which Cc_1 meets xX. But Sc . Sx = CB^2, or Sx is the polar circle of cc_1x. Hence cc_1 and Cx are conjugate lines to this circle.

FIGURE 15.

Addenda. **Proof that, when P is a point inside a circle, S and S_1 two inverse points, and XM the radical axis, then**

$$2PM \cdot FS = PK^2 + PS^2,$$

PM being the perpendicular on the radical axis and PK the semichord at right angles to FP.

Let F be the centre of the circle and S and S_1 the two inverse points which determine XM. Draw PM, PE and PK perpendicular to XM, FX, and Pf respectively;

then since $$FX^2 - SX^2 = R^2$$

and $$FX = PM + FE, \quad SX = PM - ES,$$

we have $$(PM + FE)^2 - (PM - ES)^2 = R^2$$

$$2PM(FE + ES) + FE^2 - ES^2 = R^2$$

$$2PM \cdot FS = R^2 - FE^2 + ES^2.$$

Now $$FE^2 - ES^2 = FP^2 - PS^2.$$

Substituting we get $2PM \cdot FS = R^2 - FP^2 + PS^2$

and therefore $2PM \cdot FS = PK^2 + PS^2$

which reduces to $PM \cdot FS = R \cdot PS$ when $Pf = PS$.

Fourth Meeting, 8th February 1907.

J. ARCHIBALD, Esq., M.A., President, in the Chair.

On the "e" Inequality.

By V. RAMASWAMI AIYAR, M.A.

The substance of this paper is contained in *Chrystal*, chap. XXV., §§13, and 15 to 20, with some applications thereof occurring in chap. XXVI. But it is treated here in a fresh manner which would seem simpler on several points. This mode of presentation was, in the start, suggested by Peano's method given by Prof. Gibson in his "Note on the Fundamental Inequality Theorems Connected with e^x and x^m," in Vol. XVIII. of the *Proceedings*.

1. *If x, y, z be any three positive numbers in descending order of magnitude,*

$$x^{y-z}.\,y^{z-x}.\,z^{x-y}<1.$$

Dem.: Applying the arithmetic and geometric means inequality in the form

$$a^pb^q<\left(\frac{pa+qb}{p+q}\right)^{p+q},$$

we have
$$x^{y-z}.\,z^{x-y}<\left(\frac{x(y-z)+z(x-y)}{y-z+x-y}\right)^{y-z+x-y},$$

That is,
$$x^{y-z}.\,z^{x-y}<y^{x-z}. \quad - \quad - \quad - \quad - \quad - \quad (1)$$

Hence,
$$x^{y-z}.\,y^{z-x}.\,z^{x-y}<1. \quad - \quad - \quad - \quad - \quad - \quad (2)$$

The form (2) is cyclically symmetrical, but for our applications we shall use form (1).

2. *Theorem: If $x>y>0$, then $x^{\frac{1}{x-1}}<y^{\frac{1}{y-1}}$.*

This wonderful little inequality appears to require a name; and we shall term it the "e" inequality. The name will serve to fix it well in the mind of the young reader. I have readers of *Chrystal* in view. It is supposed in the statement that neither x, nor y, is equal to 1.

Dem.: First, let $x>y>1$. Then, by inequality (1) of last article,

$$x^{y-1} . 1^{x-y} < y^{x-1}$$

$$\therefore \quad x^{\frac{1}{x-1}} < y^{\frac{1}{y-1}} \quad - \quad - \quad - \quad - \quad - \quad (a)$$

Secondly, let $x>1>y$. We have in this case

$$x^{1-y} . y^{x-1} < 1^{x-y}$$

$$\therefore \quad x^{\frac{1}{x-1}} < y^{\frac{1}{y-1}} \quad - \quad - \quad - \quad - \quad - \quad (b)$$

Thirdly, let $1>x>y$. In this case we have

$$1^{x-y} . y^{1-x} < x^{1-y}$$

which gives $$x^{\frac{1}{x-1}} < y^{\frac{1}{y-1}} \quad - \quad - \quad - \quad - \quad - \quad (c)$$

Thus the theorem holds in every case.

3. *Theorem. As x increases from* 0 *to* ∞, *the function* $x^{\frac{1}{x-1}}$ *constantly decreases from* ∞ *to* 1, *and has a definite finite limiting value*, e, *when* $x = 1$.

Dem.: The constancy of decrease follows from the inequality of the last article.

When $x=0$, the reciprocal of the function, that is, $x^{\frac{1}{1-x}}$ becomes zero; and the function therefore becomes infinite.

As x increases from 0 to 1, the function decreases in value but remains always greater than $p^{\frac{1}{p-1}}$ where p is any fixed quantity >1 chosen at pleasure; for example, it remains always greater than 2 ($p=2$). Hence $x^{\frac{1}{x-1}}$ approaches a lower limiting value A as x increases to the limit 1.

Also as x increases from 0 to 1, let y be the reciprocal of x. Then y decreases from ∞ to 1; and therefore $y^{\frac{1}{y-1}}$ constantly increases as y tends to its limit 1; but it remains always less than $q^{\frac{1}{q-1}}$ where q is any fixed positive quantity <1 chosen at pleasure; for example, it remains always less than 4 ($q=\frac{1}{2}$). Hence $y^{\frac{1}{y-1}}$ tends to an upper limit B as y decreases to the limit 1.

Now the limits A and B must be equal; for x and y being reciprocals we have, identically,

$$x^{\frac{1}{x-1}} \equiv y \,.\, y^{\frac{1}{y-1}}$$

whence $$\underset{x=1-0}{\mathrm{L}} x^{\frac{1}{x-1}} = \underset{y=1+0}{\mathrm{L}} y^{\frac{1}{y-1}}.$$

Thus we have proved that $x^{\frac{1}{x-1}}$ has a definite finite limiting value when $x=1$. This value is denoted by e.

It now remains to show that when x tends to ∞, $x^{\frac{1}{x-1}}$ tends to 1 as its limit.

We observe that $x^{\frac{1}{x-1}}$ is greater than 1 for every value of x greater than 1; for then it is a positive power of a quantity greater than 1. And it decreases as x increases. Hence when $x=\infty$, $x^{\frac{1}{x-1}}$ must have a limiting value l, which is either 1, or some definite quantity greater than 1. We can show that $l=1$ as follows:—

$$\begin{aligned} l &= \mathrm{L}x^{\frac{1}{x-1}} \quad \text{when } x=\infty\,; \\ &= \mathrm{L}\left(x^2\right)^{\frac{1}{x^2-1}}, \quad \text{changing } x \text{ into } x^2; \\ &= \mathrm{L}\left(x^{\frac{1}{x-1}}\right)^{\frac{2}{x+1}} \\ &= \left[\mathrm{L}\, x^{\frac{1}{x-1}}\right]^{\mathrm{L}\frac{2}{x+1}} \\ &= l^0 = 1. \end{aligned}$$

The proposition is thus completely proved; and the young reader will do well to fix the graph of $x^{\frac{1}{x-1}}$ unforgettably in his mind. If this be done, all that follows are simple Corollaries.

4. ***Theorem***: *If x is positive and less than* 1, $x^{\frac{1}{x-1}}>e$; *and if* $x>1$, $x^{\frac{1}{x-1}}<e$.

This follows at once from the graph of $x^{\frac{1}{x-1}}$ explained in the last article.

Cor. 1. $e > 2$.

Cor. 2. If $x > y$, then $e^x > e^y$; and if $x > y > 0$, $\log x > \log y$.
These follow from the fact that $e > 1$.

Cor. 3, 4, 5. If x is positive,

$$(1+x)^{\frac{1}{x}} < e;\quad e^x > 1 + x;\quad x > \log(1+x).$$

Cor. 6, 7, and 8. If x is positive and less than 1,

$$(1-x)^{-\frac{1}{x}} > e;\quad e^{-x} > 1 - x;\quad -x > \log(1-x).$$

5. ***Theorem***: *If x and y be positive and $x > y$,*

then
$$\frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}.$$

Dem.: Since $\frac{x}{y}$ is greater than 1,

$$\left(\frac{x}{y}\right)^{\frac{1}{\frac{x}{y}-1}} < e$$

$$\therefore\quad \left(\frac{x}{y}\right)^{\frac{1}{x-y}} < e^{\frac{1}{y}}.$$

Taking logarithms, $\dfrac{\log x - \log y}{x - y} < \dfrac{1}{y}$. . . . . . (1)

Similarly, since $\frac{y}{x}$ is positive and less than 1,

$$e^{\frac{1}{x}} < \left(\frac{y}{x}\right)^{\frac{1}{y-x}}$$

that is, $$< \left(\frac{x}{y}\right)^{\frac{1}{x-y}}$$

therefore $\dfrac{1}{x} < \dfrac{\log x - \log y}{x - y}$. . . . . . (2)

Putting (1) and (2) together, we have

$$\frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}.$$

Cor. 1. $$\frac{d}{dx}\log x = \frac{1}{x}.$$

Cor. 2. $$\underset{x=0}{\mathrm{L}} \frac{\log(1+x)}{x} = 1.$$

Cor. 3. For x, y writing e^x, e^y and taking reciprocals, we have, if $x<y$,

$$e^x > \frac{e^x - e^y}{x-y} > e^y.$$

Cor. 4. If $x \neq y$,

$$e^x(x-y) > e^x - e^y > e^y(x-y).$$

This follows from Cor. 3 if $x>y$. But it is readily seen that if the inequality is true with $x>y$, it is also true with $y>x$.

Cor. 5. In the last corollary, for x, y write $x\log a$, $y\log a$ where a is any positive quantity $\neq 1$. And we get

$$a^x\log a\,.\,(x-y) > a^x - a^y > a^y\log a\,.\,(x-y)$$

provided $x \neq y$.

Cor. 6. Hence, if $x>y$

$$a^x\log a > \frac{a^x - a^y}{x-y} > a^y\log a.$$

Cor. 7. $$\frac{d}{dx}a^x = a^x\log a.$$

Cor. 8. $$\underset{x=0}{\mathrm{L}} \frac{a^x - 1}{x} = \log a.$$

6. From the fact that $\underset{x=1}{\mathrm{L}}\, x^{\frac{1}{x-1}} = e$, we get the following corollaries:

(1) $$\underset{x=0}{\mathrm{L}}\, (1+x)^{\frac{1}{x}} = e.$$

(2) $$\underset{n=\infty}{\mathrm{L}} \left(1 + \frac{x}{n}\right)^n = e^x.$$

(3) If nu_n tends to the limit a when $n=\infty$,

$$\underset{n=\infty}{\mathrm{L}}\, (1+u_n)^n = e^a.$$

(4) In particular, if $\mathrm{L}nu_n = 0$ when $n=\infty$,

$$\underset{n=\infty}{\mathrm{L}}\, (1+u_n)^n = 1.$$

7. The fact that $\underset{x=\infty}{\mathrm{L}}\, x^{\frac{1}{x-1}} = 1$ leads to a number of important limit theorems to which numerous others are reducible.

(1) $\underset{x=\infty}{\mathrm{L}}\, x^{\frac{1}{x}} = 1$. (Standard theorem for the form ∞^0.)

(2) $\underset{x=+0}{\mathrm{L}}\, x^x = 1$. (Standard theorem for the form 0^0.) This is deducible from (1) by writing $\dfrac{1}{x}$ for x.

(3) $\underset{x=\infty}{\mathrm{L}}\, \dfrac{\log x}{x} = 0$, which is deduced from (1) by taking logarithms. This is a standard theorem, explaining the logarithmic scale.

(4) $\underset{x=\infty}{\mathrm{L}}\, \dfrac{e^x}{x} = \infty$. This follows from (3) by writing e^x for x.

(5) $\underset{x=+0}{\mathrm{L}}\, x\log x = 0$. This follows from (2) by taking logarithms.

The Higher Logarithmic Inequalities.

8. Let lx, l^2x, ... denote $\log x$, $\log\log x$, ... Then, if x and y be positive and $x > y$, and y sufficiently great to render $l^r y$ positive,

$$\frac{1}{x \,.\, lx \,.\, l^2x \ldots l^rx} < \frac{l^{r+1}x - l^{r+1}y}{x-y} < \frac{1}{y \,.\, ly \,.\, l^2y \ldots l^ry}.$$

Proof: We have, by Art. 5,

$$\frac{1}{x} < \frac{lx - ly}{x-y} < \frac{1}{y}$$

$$\frac{1}{lx} < \frac{l^2x - l^2y}{lx - ly} < \frac{1}{ly}$$

$$\frac{1}{l^rx} < \frac{l^{r+1}x - l^{r+1}y}{l^rx - l^ry} < \frac{1}{l^ry}.$$

Multiplying all these together we have the result given

9. If x and y be positive and $x>y$, and y sufficiently great to render $l^r y$ positive, and β be any positive quantity

$$\frac{\beta}{x \,.\, lx \ldots l^{r-1}x (l^r x)^{1+\beta}} < \frac{(l^r y)^{-\beta} - (l^r x)^{-\beta}}{x - y} < \frac{\beta}{y \,.\, ly \ldots l^{r-1}y \,.\, (l^r y)^{1+\beta}}.$$

Proof: By the power inequality we have if $X > Y > 0$

$$-\beta X^{-\beta-1} > \frac{X^{-\beta} - Y^{-\beta}}{X - Y} > -\beta \,.\, Y^{-\beta-1}.$$

Multiplying this by -1, we have

$$\frac{\beta}{X^{1+\beta}} < \frac{Y^{-\beta} - X^{-\beta}}{X - Y} < \frac{\beta}{Y^{1+\beta}}.$$

Now for X, Y write $l^r x$, $l^r y$; and we have

$$\frac{\beta}{(l^r x)^{1+\beta}} < \frac{(l^r y)^{-\beta} - (l^r x)^{-\beta}}{l^r x - l^r y} < \frac{\beta}{(l^r y)^{1+\beta}}. \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

And by the theorem of the last article,

$$\frac{1}{x \,.\, lx \ldots l^{r-1}x} < \frac{l^r x - l^r y}{x - y} < \frac{1}{y \,.\, ly \ldots l^{r-1}y}. \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

Multiplying (1) and (2) we have the theorem given.

Application to Infinite Series.

10. By the power inequality, which, it may be remarked, is an equivalent of the "e" inequality, we have, if β be any positive quantity,

$$\frac{\beta}{(n+1)^{1+\beta}} < n^{-\beta} - (n+1)^{-\beta} < \frac{\beta}{n^{1+\beta}}.$$

From this we can readily deduce that the series $\sum_1^\infty \frac{1}{n^{1+\beta}}$ is convergent, $(\beta > 0)$.

The power inequality also gives, if β be positive and less than 1,

$$\frac{\beta}{(n+1)^{1-\beta}} < (n+1)^\beta - n^\beta < \frac{\beta}{n^{1-\beta}}.$$

From this it is deducible that the series $\sum_1^\infty \frac{1}{n^{1-\beta}}$ is divergent, (β positive and < 1).

If β be not less than 1, the divergency of $\sum\limits_1^\infty \dfrac{1}{n^{1-\beta}}$ is at once apparent.

The divergency of $\sum\limits_1^\infty \dfrac{1}{n}$ can be similarly deduced from the logarithmic inequality of Art. 5.

It can be similarly proved from the inequality of Art. 8, that the series $\sum\limits_p^\infty \dfrac{1}{n\,.\,ln\,.\,l^2n\,\ldots\,l^rn}$ is divergent.

It follows that the series

$$\sum_p^\infty \frac{1}{n\,.\,ln\,\ldots\,l^{r-1}n(l^rn)^{1-\beta}},$$

where β is any positive quantity is divergent.

The inequality of Art. 9 similarly gives us that the series

$$\sum_p^\infty \frac{1}{n\,.\,ln\,\ldots\,l^{r-1}n(l^rn)^{1+\beta}}$$

is convergent, if β is any positive quantity.

11. These are the standard theorems for comparison in determining the absolute convergence, or otherwise, of series whose ratio of convergence tends to the limit 1. They are here shown to be deducible without Cauchy's Condensation Test. By the method here indicated we could, further, assign upper and lower limits to the remainder after n terms in all cases where the series is convergent. Even where the series diverges, we derive valuable information regarding the nature of the divergency. Thus, in the case of $\Sigma \dfrac{1}{n}$ we can see by the inequality of Art. 5, that

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n$$

tends to a definite finite limit, the Eulerian Constant γ, when n becomes infinite.

Equivalents of the "e" Inequality.

12. We remarked that the power inequality was an equivalent of the "e" inequality. More fully, it is worth noting that the following *six* inequalities are equivalent to one another.

I. $$\frac{pa+qb}{p+q} > (a^p b^q)^{\frac{1}{p+q}}$$

if a, b, p, q are positive and $a \neq b$.

II. $$a^x(y-z)+a^y(z-x)+a^z(x-y)>0$$

if x, y, z are in descending order of magnitude and a is any positive quantity $\neq 1$.

III. $$\frac{a^x-1}{x} > \frac{a^y-1}{y},$$

if $x>y$ and a is any positive quantity $\neq 1$.

II′. If $$x>y>z>0,$$
then $$x^{y-z}.\,y^{z-x}.\,z^{x-y}<1.$$

III′. The "e" inequality.

IV. The power inequality:—If a is positive and not equal to 1.
$$a^m-1 \lesseqgtr m(a-1) \text{ according as } m(m-1) \lesseqgtr 0.$$

Arranging these six inequalities in circular order, we can from each inequality deduce the one following it, proceeding in the clockwise order, or counter clockwise order, at pleasure. I shall leave the details to the reader; but shall here indicate how the power inequality can be deduced from the "e" inequality, by proving IV. in one case, that is, when $a>1$, and $m>1$. In this case we have
$$a<1+m(a-1).$$
Consequently, by the "e" inequality
$$a^{\frac{1}{a-1}} > [1+m(a-1)]^{\frac{1}{m(a-1)}}.$$

Raising both sides to the power $m(a-1)$ which is positive, and transposing 1,
$$a^m-1>m(a-1).$$

Note on The Envelope-Investigation.

By D. K. Picken, M.A.

Obscurity in the direct discussion of The Envelope, as given in works on Differential Equations, has led writers on The Calculus to define the envelope of a family by a property which all know that it shares with any locus of multiple-points belonging to the family. The following presentation is an attempt by use of systematic notation to make clear the details of the direct process :—

Starting from the definition that

A curve is an envelope of a given family, if at each of its points it touches a member of the family :

let us suppose that a family is specified by the equation

$$\psi(x, y, u) = 0$$

in which ψ is a continuous function of *the three variables* x, y, u; continuous variation of u corresponds to continuous motion and deformation of a variable curve in the xy-plane, which takes in succession the curves of the family as positions.

If there is an envelope, it is the locus of a variable point P_u which has for a given value (u_0) of u, a *determinate* position (P_{u_0}) on the u_0-curve of the family: the coordinates (x_u, y_u) of P_u are therefore unknown functions of u such that

$$\psi(x_u, y_u, u) \equiv 0, \qquad - \quad - \quad - \quad - \ (1)$$

and we seek to determine x_u, y_u from the fact that the locus of P_u touches the u_0-curve at the point P_{u_0}. This property gives the identical relation

$$\frac{dy_u}{du}\bigg/\frac{dx_u}{du} \equiv -\psi_x'(x_u, y_u, u)\Big/\psi_y'(x_u, y_u, u)$$

i.e.,
$$\psi_x'(x_u, y_u, u)\,.\,\frac{dx_u}{du} + \psi_y'(x_u, y_u, u)\,.\,\frac{dy_u}{du} \equiv 0. \qquad - \ (2)$$

But from the identity (1),

$$\psi_x'(x_u,\ y_u,\ u)\,.\,\frac{dx_u}{du}+\psi_y'(x_u,\ y_u,\ u)\frac{dy_u}{du}+\psi_u'(x_u,\ y_u,\ u)\equiv 0\,;$$

hence (1) and (2) are equivalent to

$$\psi(x_u,\ y_u,\ u)\equiv 0 \text{ and } \psi_u'(x_u,\ y_u,\ u)\equiv 0.$$

Any envelope-locus is, therefore, represented in the relation between x and y which is the eliminant of u from the equations

$$\psi(x,\ y,\ u)=0,\ \ \psi_u'(x,\ y,\ u)=0.$$

The full locus of this eliminant-equation may be geometrically described as

(i) the locus of ultimate points of intersection of "consecutive" curves of the family; and therefore as

(ii) the locus of points in the xy-plane at which the equation $\psi(x,\ y,\ u)=0$ has two roots in u equal;

it obviously includes, on the description (i), the envelopes and multiple-point loci; and the description (ii) of this locus leads to the conclusion that the envelopes are also included in the "p-discriminant" of the differential equation that represents the family.

On the teaching to beginners of such transformations as $-(-a)=+a$.

By D. C. M'Intosh, M.A.

Fifth Meeting, 8th March 1907.

J. Archibald, Esq., M.A., President, in the Chair.

On Area-Theory, and some applications.

By P. Pinkerton, M.A.

1. In the *Cambridge and Dublin Mathematical Journal*, vol. v., 1859, De Morgan gives the definition of the "area contained within a circuit" as the area swept out by a radius vector which has one end (the pole) fixed and the other describing the circuit (in a determinate mode), on the supposition that each element of area is positive or negative, according as the radius is revolving positively or negatively. He remarks that the definition satisfies existing notions, that it provides the necessary extension of the meaning of the word area, and proceeds to show that it gives to every circuit the same area, whatever point the pole may be. The object of this paper is to give an Area-Theory beginning with the triangle and going on to circuits bounded by straight or curved lines. The fundamental proposition is derived from Analysis, and the geometry of the applications is therefore an Analytical Geometry; indeed, one of the objects of the paper is to emphasise the advantage of keeping Analysis and Geometry in close correspondence. As evidence of the difficulty of pursuing an Area-Theory in Geometry, without the aid of Analysis, it may be noticed that Townsend in his *Modern Geometry* (1863), § 83, lays down Salmon's Theorem in this form: "If A, B, C, D be any four points on a circle taken in the order of their disposition, and P any fifth point, without, within, or upon the circle, but not at infinity, then always

$$\text{area } BCD.AP^2 - \text{area } CDA.BP^2 + \text{area } DAB.CP^2 - \text{area } ABC.DP^2 = 0,$$

regard being had only to the absolute magnitudes of the several areas which from their disposition are incapable of being compared in sign." Yet, previous to this, he uses positive and negative area of the triangle; and, later on (Chap. vii), works out at some length a formal definition of the "area of a polygon," "whether convex, reentrant, or intersecting."

2. Let (x_1, y_1) and (x_2, y_2) be the coordinates of points P_1, P_2 with reference to a rectangular Cartesian system of reference, origin O; to find an expression for the measure of $\triangle OP_1P_2$ in terms of x_1, y_1, x_2, y_2.

Let (r_1, θ_1) and (r_2, θ_2) be polar coordinates of P_1, P_2 with reference to O as pole and OX as initial line; r_1, r_2 being positive, and θ_1, θ_2 being *any* angles through which OX must turn to come into the positions OP_1, OP_2. Let $P_1\widehat{O}P_2$ be the angle through which OP_1 must turn to come into the position OP_2, under the condition that the radius vector traces out the angle O of the triangle P_1OP_2; then $P_1\widehat{O}_2P$ has sign as well as magnitude.

$$\text{Then} \qquad \theta_1 + P_1\widehat{O}P_2 = 2n\pi + \theta_2 \ (n \text{ integral or zero});$$
$$\therefore \ P_1\widehat{O}P_2 = 2n\pi + (\theta_2 - \theta_1);$$
$$\therefore \ \sin P_1OP_2 = \sin(\theta_2 - \theta_1),$$

and is positive or negative according as OP_1P_2O indicates the trigonometrically positive sense or the trigonometrically negative sense of rotation in the plane.

Now the absolute measure of $\frac{1}{2}r_1r_2\sin P_1OP_2$ is the area of triangle OP_1P_2; we introduce positive and negative area by *defining* $\frac{1}{2}r_1r_2\sin P_1OP_2$ *or* $\frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$ *as the measure of* $\triangle OP_1P_2$, and write

$$\triangle OP_1P_2 = \tfrac{1}{2}r_1r_2\sin(\theta_2 - \theta_1) = \tfrac{1}{2}(x_1y_2 - x_2y_1),$$
$$\text{and} \quad \triangle OP_2P_1 = \tfrac{1}{2}r_2r_1\sin(\theta_1 - \theta_2) = \tfrac{1}{2}(x_2y_1 - x_1y_2).$$

The *sign* of the expression $\frac{1}{2}(x_1y_2 - x_2y_1)$ has a specific geometrical meaning, and the order of the letters OP_1P_2 has a corresponding significance.

If A, B, C are three points in a plane, we say that $\triangle ABC$ is "a positive area" or "a negative area," according as the sequence of letters ABCA indicates the positive or negative sense of circulation in the plane, as already agreed on in Trigonometry.

3. To find an expression for the measure of $\triangle P_1P_2P_3$ in terms of the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) of three points P_1, P_2, P_3 in the plane of the axes.

$\triangle P_1P_2P_3$, that is, $\frac{1}{2}P_1P_2 . P_1P_3\sin P_2P_1P_3$, is unaltered by change of axes. Change to parallel axes through the point (x_1, y_1). Let (ξ_2, η_2), (ξ_3, η_3) be the new coordinates of P_2, P_3; then

$$\triangle P_1P_2P_3 = \tfrac{1}{2}(\xi_2\eta_3 - \xi_3\eta_2) = \tfrac{1}{2}\{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\}$$
$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}.$$

4. **From** § 3 comes the general Area-theorem,

$$\triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1,$$

connecting the areas (regarded as having *sign*) associated with any four coplanar points.

Cor. 1. The relation can be more systematically expressed thus: for any four coplanar points $P_1P_2P_3P_4$

$$\triangle P_2P_3P_4 - \triangle P_3P_4P_1 + \triangle P_4P_1P_2 - \triangle P_1P_2P_3 = 0.$$

Cor. 2. If A, B, C, ... K, L are collinear points, and O any other point

$$\triangle OAL = \triangle OAB + \triangle OBC + \ldots + OKL.$$

5. This theorem may be regarded as proving that if P_1, P_2, P_3 *are fixed points, and* Q *a variable point of their plane*

$$(\triangle QP_1P_2 + \triangle QP_2P_3 + \triangle QP_3P_1)$$

does not vary with Q.

The theorem in this form has the following important extension: *If* P_1, P_2, ..., P_n *are any n given coplanar points, and* Q *a variable point of their plane,* $(\triangle QP_1P_2 + \triangle QP_2P_3 + \ldots + \triangle QP_{n-1}P_n + \triangle QP_nP_1)$ *does not vary with* Q.

Proof. If O is any base-point of the plane,

$$\begin{aligned} \triangle QP_rP_{r+1} &= \triangle OQP_r + \triangle OP_rP_{r+1} + \triangle OP_{r+1}Q, \\ &= \triangle OP_rP_{r+1} + \triangle OQP_r - \triangle OQP_{r+1}. \end{aligned}$$

$\therefore \quad \Sigma\triangle QP_rP_{r+1} = \Sigma OP_rP_{r+1}$, for a complete cycle.

6. Now consider a simple closed plane space bounded by straight lines P_1P_2, P_2P_3 ..., $P_{n-1}P_n$, P_nP_1 in order and first suppose the boundary is *convex*. Give Q a position within the boundary. Then $(\triangle QP_1P_2 + \triangle QP_2P_3 + \ldots + \triangle QP_nP_1)$ is in absolute measure the area* of the closed space. Therefore the absolute measure of the same expression is the area* of the closed space, for *all* positions of Q.

Next suppose that the boundary is not convex. Break the area*

* "Area" here means simply area, and is of course neither positive nor negative.

of the closed space into areas* of simple closed spaces with convex boundaries by introducing cross-lines such as P_rP_s in fig. 18. Then

$$(\Delta QP_1P_2 + \Delta QP_2P_3 + \ldots + \Delta QP_nP_1)$$
$$= \{\Delta QP_1P_2 + \ldots + \Delta QP_nP_1 + \Sigma(\Delta QP_rP_s + \Delta QP_sP_r)\}$$

= $\pm$ sums of areas* of closed spaces with convex boundaries, since each of these areas* would appear with the same sign prefixed.

Hence again

absolute measure of $(\Delta QP_1P_2 + \Delta QP_2P_3 + \ldots + \Delta QP_nP_1)$
= area* of closed space.

Hence for the most general coplanar positions of P_1, P_2, ..., P_n, we *define area* $P_1P_2 \ldots P_nP_1$ *to be*

$$(\Delta QP_1P_2 + \Delta QP_2P_3 + \ldots + \Delta QP_{n-1}P_n + \Delta QP_nP_1),$$

Q *being any coplanar point.*

7. Any one of the lines P_1P_2, P_2P_3, ..., P_nP_1, supposed terminated at the extremities P_1, P_2; etc., may now cross any other. Consider fig. 19. Each of the lines P_1P_2, etc., crosses two or more of the others. Mark the crossing-points as in the figure. Then

$$\Delta QP_1P_2 = \Delta QP_1R_1 + \Delta QR_1R_2 + \Delta QR_2P_2,$$
$$\Delta QP_2P_3 = \Delta QP_2R_3 + \Delta QR_3R_4 + \Delta QR_4R_5 + \Delta QR_5P_3,$$
etc., etc.,

$\therefore$ Area $P_1P_2 \ldots P_6P_1$

$$= \text{Area } P_1R_1R_4R_5P_1 + \text{Area } P_2R_3R_2P_2 + \text{Area } P_3R_6R_5P_3$$
$$+ \text{Area } P_4R_3R_4R_7P_4 + \text{Area } P_5R_1R_2P_5 + \text{Area } P_6R_6R_7P_6.$$

In estimating Area $P_1R_1R_4R_5P_1$, etc., give Q a position within each boundary in turn, and the signs of these partial areas are seen to be, in order, +, +, −, −, −, +. This result corresponds to De Morgan's Rule for Area.

The following sections contain some applications of the above theory.

8. Note (i) that ΔQ_1AB, ΔQ_2AB are of the same or of opposite sign according as Q_1, Q_2 are on the same or on opposite sides of the AB-line.

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* "Area" means simply area, and of course is neither positive nor negative.

(ii) that if AB, CD are steps on the same line, $\triangle QAB$ and $\triangle QCD$ are of the same or of opposite signs according as AB, CD are steps of the same or of opposite sign.

Hence the fundamental theorem

$$\triangle QAB : \triangle QCD = AB : CD$$

is to be regarded as taking account of *sign*.

In particular, if M is the middle point of AB, $\triangle QAM = \triangle QMB$.

Euc. VI., 2 can be written out in such a way as to suit all figures. Let B_1C_1 parallel to base BC of triangle ABC meet the lines AB, AC in B_1, C_1 respectively. Then B_1, C_1 are on the same side of BC,

$$\therefore \quad \triangle BCC_1 = \triangle BCB_1,$$

$$\therefore \quad \triangle ABC + \triangle ACC_1 + \triangle AC_1B = \triangle ABC + \triangle ACB_1 + \triangle AB_1B,$$

$$\therefore \quad \triangle AC_1B = \triangle ACB_1, \text{ since } \triangle ACC_1 = 0 = \triangle AB_1B.$$

Hence $\quad AB : AB_1 = \triangle ABC : \triangle AB_1C = \triangle ABC : \triangle ABC_1$

$$= AC : AC_1.$$

Again, a direct and general proof of Ceva's Theorem can be given.

Let concurrent lines AOD, BOE, COF meet the sides BC, CA, AB of triangle ABC in D, E, F respectively.

$$\begin{aligned} BD : CD = \triangle OBD : \triangle OCD &= \triangle ABD : \triangle ACD \\ &= \triangle OAB + \triangle OBD : \triangle OAC + \triangle OCD, \\ &\qquad \text{since } \triangle ODA = 0 \\ &= -(\triangle OAB : \triangle OCA). \end{aligned}$$

Similarly $\quad CE : AE = -(\triangle OBC : \triangle OAB),$

$$AF : BF = -(\triangle OCA : \triangle OBC)$$

$$\therefore \quad \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1.$$

(iii) $\triangle Q_1AB : \triangle Q_2AB = p_1 : p_2$,

where p_1, p_2 are the *ordinates* of Q_1, Q_2 with respect to the AB-line, in other words the *perpendiculars* from Q_1, Q_2 to the AB-line, if the *perpendiculars* are regarded as *steps*.

This may be shown by taking A, B as points on the x-axis of a system of Rectangular axes and applying the formula for $\triangle P_1P_2P_3$ in terms of the coordinates of P_1, P_2, P_3.

9. If A, B, C, O are any four points in a plane and G the middle point of BC, then

$$\triangle OAB + \triangle OAC = 2\triangle OAG.$$

For $\triangle OAB + \triangle OBG + \triangle OGA = \triangle ABG = \triangle AGC$

$$= \triangle OAG + \triangle OGC + \triangle OCA$$

$\therefore\ \triangle OAB + \triangle OAC = 2\triangle OAG$, since $\triangle OBG = \triangle OGC$.

Hence, if M is the middle point of AB, P and Q two other points of the plane

$$\triangle APQ + \triangle BPQ = 2\triangle MPQ,$$

being a form of $\triangle PQA + \triangle PQB = 2\triangle PQM$.

And again, if M is half-way from A to the PQ-line,

$$\triangle APQ = 2\triangle MPQ.$$

10. If A, B, C, D are any four points of a plane; E, F, G, H the middle points of AB, BC, CD, DA respectively, then

$$\text{Area EFGH} = \tfrac{1}{2}\ \text{Area ABCD}.$$

Area EFGH $= \triangle AEF + \triangle AFG + \triangle AGH + \triangle AHE$

$\triangle AEF = \tfrac{1}{2}\triangle ABF = \tfrac{1}{4}\triangle ABC$,

$\triangle AFG = \tfrac{1}{2}(\triangle AFC + \triangle AFD) = \tfrac{1}{4}(\triangle ABC + \overline{\triangle ABD + \triangle ACD})$,

$\triangle AGH = \tfrac{1}{2}\triangle AGD = \tfrac{1}{4}\triangle ACD$,

$\triangle AHE = \tfrac{1}{2}\triangle ADE = \tfrac{1}{4}\triangle ADB$,

$\therefore$ Area EFGH $= \tfrac{1}{2}(\triangle ABC + \triangle ACD) = \tfrac{1}{2}$ Area ABCD.

11. If A, B, C, D are any four points of a plane, P and Q the middle points of AC, BD respectively, X the point of intersection of the AD- and the BC-lines, Y the point of intersection of the AB- and CD-lines, then

$$\triangle XPQ = \tfrac{1}{4}\ \text{Area ABCD},$$

$$\text{and}\quad \triangle YPQ = -\tfrac{1}{4}\ \text{Area ABCD}.$$

$2\triangle XPQ = \triangle XPD + \triangle XPB$,

$= \tfrac{1}{2}\triangle XCD + \tfrac{1}{2}\triangle XAB$,

$= \tfrac{1}{2}(\triangle XAB + \triangle XBC + \triangle XCD + \triangle XDA)$,

since $\triangle XBC = 0 = \triangle XDA$.

$= \tfrac{1}{2}$ Area ABCD.

Similarly $\qquad \triangle YPQ = -\frac{1}{4}$ Area ABCD.

Cor. Hence $\qquad \triangle XPQ + \triangle YPQ = 0$

therefore the middle point of XY is on the PQ-line, *i.e.*, the middle points of the diagonals of a complete quadrilateral are collinear.

12. If A, B, C, D are any four given points of a plane, and if a variable point P moves so that

$$m \,.\, \triangle PAB + n \,.\, \triangle PCD = \text{constant},$$

when m, n are any fixed multiples positive or negative, then the locus of P is a straight line.

Let the AB- and CD-lines meet in O. Let $OX = m \,.\, AB$ and $OY = n \,.\, CD$ in sign and magnitude, and let G be the middle point of XY.

Then
$$\begin{aligned} m \,.\, \triangle PAB + n \,.\, \triangle PCD &= \triangle POX + \triangle POY \\ &= 2\triangle POG. \end{aligned}$$

$\therefore$ locus of P is a straight line parallel to OG.

An obvious extension is that if A_1B_1, A_2B_2, ..., A_nB_n are n fixed lines in a plane, and P a variable point such that

$$a_1 \,.\, \triangle PA_1B_1 + a_2 \,.\, \triangle PA_2B_2 + \dots + a_n \,.\, \triangle PA_nB_n = \text{constant},$$

where a_1, a_2, ..., a_n are fixed multiples, positive or negative, then the locus of P is a straight line.

Cor. An equation of the first degree in areal coordinates represents a straight line.

13. The following problem illustrates the use of the theory geometrically.

Let A, B be two fixed points in a plane, C, D two variable points in the plane, such that CD is fixed in magnitude and direction and Area ABCD is fixed; to find the loci of C and D.

Draw AE parallel to CD such that $AE = DC$, in sign and magnitude.

Then Area $ABCD = \triangle ABE + \triangle CEB + \triangle CDAE$,

$$\therefore \text{ Area } ABCD - \triangle ABE$$
$$= \triangle CEB + 2\triangle CAE$$
$$= \triangle CEB + \triangle CEF, \text{ where } AE \text{ is produced to } F$$
$$\text{so that } EF = 2AE \text{ in sign and magnitude}$$
$$= 2\triangle CEG, \text{ if } G \text{ is the middle point of } BF;$$

therefore $\triangle CEG$ is constant. Hence the locus of C is a straight line parallel to EG, and therefore the locus of D is a parallel straight line, since CD is fixed in magnitude and direction.

14. If A, B, C, O are any four coplanar points, the mean centre of the points A, B, C for multiples $\triangle OBC$, $\triangle OCA$, $\triangle OAB$, or multiples proportional to these, is the point O.

For
$$\frac{\triangle OBC}{\triangle OCA} = -\frac{\triangle BCO}{\triangle ACO} = -\frac{b}{a},$$

where b, a are the perpendiculars from B, A to OC, account being taken of sign.

$$\therefore\ a \,.\, \triangle OBC + b \,.\, \triangle OCA = 0$$

and hence OC passes through the mean centre of ABC for multiples $\triangle OBC$, $\triangle OCA$, OAB.

Similarly OA, OB pass through the mean centre for those multiples. Therefore O is the mean centre.

Hence if A, B, C, D be any four points on a circle, and O any fifth point in the plane

$$OB^2.\triangle ACD + OC^2.\triangle ADB + OD^2.\triangle ABC - (\triangle ABC + \triangle ACD + \triangle ADB)OA^2$$
$$= AB^2.\triangle ACD + AC^2.\triangle ADB + AD^2.\triangle ABC$$
$$= \text{constant, for all positions of } O.$$

Giving O the position of the centre of the circle, and noting that

$$\triangle ABC + \triangle ACD + \triangle ADB = \triangle BCD, \quad \text{we see that}$$

$$OA^2.\triangle BCD - OB^2\triangle CDA + OC^2\triangle DAB - OD^2.\triangle ABC = 0,$$

and
$$AB^2.\triangle ACD + AC^2\triangle ADB + AD^2\triangle ABC = 0.$$

Also, if x, y, z are the areal coordinates of any point P on a circle and ABC, the triangle of reference, taking P, A, B, C in turn as mean centres of A, B, C; B, C, P; etc.; we have

$$x\,.\,\mathrm{PA}^2+y\,.\,\mathrm{PB}^2+z\,.\,\mathrm{PC}^2=0,$$

$$\mathrm{PA}^2=yc^2+zb^2,$$

$$\mathrm{PB}^2=za^2+xc^2,$$

$$\mathrm{PC}^2=xb^2+ya^2,$$

whence $$yza^2+zxb^2+xyc^2=0.$$

15. In extending the theory to areas of closed plane spaces bounded by curves or partly bounded by curves, the following Lemma is useful:

If $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$ *be any n given points of a plane,* $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n$ *n collinear points of the plane*

$$\textit{Area}\ \mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_n\mathrm{P}_1=\textit{Area}\ \mathrm{P}_1\mathrm{P}_2\mathrm{L}_2\mathrm{L}_1+\ldots+\textit{Area}\ \mathrm{P}_r\mathrm{P}_{r+1}\mathrm{L}_{r+1}\mathrm{L}_r+\ldots$$
$$\ldots+\textit{Area}\ \mathrm{P}_n\mathrm{P}_1\mathrm{L}_1\mathrm{L}_n.$$

For

$$\text{Area}\ \mathrm{P}_r\mathrm{P}_{r+1}\mathrm{L}_{r+1}\mathrm{L}_r=\Delta\mathrm{OP}_r\mathrm{P}_{r+1}-\Delta\mathrm{OL}_r\mathrm{L}_{r+1}-(\Delta\mathrm{OP}_r\mathrm{L}_r-\Delta\mathrm{OP}_{r+1}\mathrm{L}_{r+1})$$

and $$\Sigma\Delta\mathrm{OL}_r\mathrm{L}_{r+1}=0,\ \Sigma(\Delta\mathrm{OP}_r\mathrm{L}_r-\Delta\mathrm{OP}_{r+1}\mathrm{L}_{r+1})=0.$$

If $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n$ be the projections $\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n$ of $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$ on the x-axis of a rectangular Cartesian system,

$$\begin{aligned}\text{Area}\ \mathrm{P}_1\mathrm{P}_2\mathrm{M}_2\mathrm{M}_1&=\Delta\mathrm{OP}_1\mathrm{P}_2+\Delta\mathrm{OP}_2\mathrm{M}_2+\Delta\mathrm{OM}_2\mathrm{M}_1+\Delta\mathrm{OM}_1\mathrm{P}_1\\&=\tfrac{1}{2}(x_1y_2-x_2y_1-x_2y_2+x_1y_1)\\&=-\tfrac{1}{2}(x_2-x_1)(y_1+y_2).\end{aligned}$$

If $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n$ be the projections $\mathrm{N}_1, \mathrm{N}_2, \ldots, \mathrm{N}_n$ of $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$ on the y-axis,

$$\text{Area}\ \mathrm{P}_1\mathrm{P}_2\mathrm{N}_2\mathrm{N}_1=\tfrac{1}{2}(x_1+x_2)(y_2-y_1).$$

16. If $\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_n\mathrm{P}_1$ specifies the boundary of a closed curve, the area of the space enclosed is defined to be

$$\underset{n=\infty}{\mathrm{Lt}}\ \Sigma(\Delta\mathrm{QAP}_1+\Delta\mathrm{QP}_1\mathrm{P}_2+\ldots+\Delta\mathrm{QP}_n\mathrm{A}),$$

where A is a fixed point on the curve and the P's are distributed on

the curve according to some law such that $\underset{n \to \infty}{\text{Lt}}\ P_rP_{r+1} = 0$ and that a current point P moving steadily round the curve from A to A passes through $P_1, P_2, \ldots, P_n$ in succession.

Hence $\text{Area} = \frac{1}{2}\int r^2 d\theta$ from $\text{Lt}\ \Sigma r(r+\Delta r)\sin\Delta\theta$,

and $\text{Area} = \frac{1}{2}\int (xdy - ydx)$ from $\text{Lt}\ \Sigma \frac{1}{2}\{x(y+\Delta y) - (x+\Delta x)y\}$.

Again, from the expressions for Area $P_1P_2M_2M_1$ and Area $P_1P_2N_2N_1$ in § 15 it is clear that

$$\text{Area} = -\int ydx = \int xdy.$$

If HA, KB are ordinates of A, B two points on a curve represented by the equation $y = f(x)$, where $f(x)$ is a single-valued continuous function of x, then along AH and BK, $dx = 0$; and along HK, $y = 0$. Therefore Area $\text{AHKB} = \int_a^b ydx$, where a, b are the abscissae of A, B.

And if P is a variable point (x, y) on the curve and MP its ordinate

$$\frac{dA}{dx} = y$$

where A = Area AHMP.

For take Q a point on the curve near to P, then

$$A + \Delta A = \text{Area AHNQ} = \text{Area AHMP} + \text{Area PMNQ}$$

$$\therefore\ \Delta A = \text{Area PMNQ} = +y\Delta x$$

$$\therefore\ \frac{dA}{dx} = y.$$

If a new variable t be introduced where x, y are single-valued functions of t, and t varies always in one sense (that is, always increasing or always decreasing) from t_1 to t_2 as the current point P moves round the curve from A to A, passing through $P_1, P_2, \ldots,$ in succession, we have formulæ such as

$$\text{Area} = \frac{1}{2}\int_{t_1}^{t_1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$$

It is sometimes said that t must be chosen so as to go on *increasing* when the current point P moves steadily round the boundary *leaving the area on the left*. There are two misleading

elements in such a statement. First, t may go on ***decreasing or increasing***. Secondly, in cases where the boundary crosses itself, it is not possible for the current point P to move steadily round the boundary and ***always*** leave the ***area*** on left or right.

For example, in fig. 20

$$\tfrac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt = \text{Area } AP_1P_2 \ldots P_5A$$

$$= \text{area* of space (2)} - \text{area* of space (1)},$$

space (2) being to ***left*** of current point, while space (1) is to ***right*** of current point.

In fig. 21

$$\text{Integral} = \text{Area} AP_1P_2 \ldots P_5A$$

$$= \text{twice area* of space (1)} + \text{area* of space (2)},$$

space (2) not including the shaded portion.

In fig. 22

$$\text{Integral} = \text{Area } AP_1P_2 \ldots P_{12}A$$

$$= \text{area* of shaded space} + \text{twice area* of space (4)}$$

$$- \text{sum of areas* of spaces (1), (2), (3)}.$$

It is worth noting that, using double integrals, we have

$$\iint dx\,dy = \tfrac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt,$$

the simplest case of Stokes's Theorem.

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* "Area" being here neither positive nor negative.

On Commutative Matrices.

By J. H. Maclagan-Wedderburn, M.A.

Sixth Meeting, 10th May 1907.

J. ARCHIBALD, Esq., M.A., President, in the Chair.

On Certain Projective Configurations in Space of n Dimensions and a Related Problem in Arrangements.

By D. M. Y. SOMMERVILLE, M.A., D.Sc.

In the first part of this paper there are found the numbers of points, lines, etc., in a finite projective geometry of n dimensions. The substance of this has already been worked out by O. Veblen and W. H. Bussey.* The second part is concerned with the arrangements of the numbers representing the points in a finite projective plane desarguesian geometry.

I.

1. Consider an assemblage of points, lines, planes, 3-spaces, ... $(n-1)$-spaces in space of n dimensions R_n. With regard to these we shall make the assumptions :

(A) An $(r-1)$-space and a point belonging to the system and not in the $(r-1)$-space always determine an r-space belonging to the system.

(B) In an R_t a line cuts an R_{t-1} in a point belonging to the system.

From these assumptions it follows that

(A)′ an R_r and an R_s which both pass through an R_t $(r, s > t)$ always determine an R_{r+s-t} belonging to the system and containing the R_r and the R_s.

(B)′ in an R_t every R_s cuts every R_r $(r+s \geqq t)$ in an R_{r+s-t} which belongs to the system.

* "Finite projective geometries," *Amer. M. S. Trans.*, vii. (1906), 241–259. See also Whitehead, "The axioms of projective geometry," p. 13.

We shall further assume

(C) that on each line there are the same number of points.

From (A), (B), and (C) it will be shown that

(C)′ in each R_t there are the same number of R_r's $(r<t)$; and through each R_s and lying in an R_t containing R_s there pass the same number of R_r's $(t>r>s)$; for all values of r, s, t subject to the given conditions. These numbers will all be expressed in terms of the number of points in a line.

2. Let the number of r-spaces which pass through, or lie in, an s-space (according as $r \gtrless s$) be denoted by p_{rs}; and the number of r-spaces containing a given s-space and contained in a given t-space be denoted by (r, s, t). In this notation $t>r>s$. If this condition is not satisfied, then for (r, s, t) we must substitute another expression, viz.,

$$(r, t, s) \text{ if } s>r>t;$$

$$p_{rs} \text{ if } t>s>r \text{ or } t<s<r;$$

$$p_{rt} \text{ if } s>t>r \text{ or } s<t<r.$$

Also if

$$s=t, \qquad (r, s, t)=p_{rs}=p_{rt};$$

$$r=s,\ t \text{ or } n, \quad (r, s, t)=1;$$

$$t=n, \qquad (r, s, t)=p_{rs},$$

$$s=n, \qquad (r, s, t)=p_{rt}.$$

3. *To express (r, s, t) in terms of the p's.*

In an R_t there are p_{rt} r-spaces; on each there lie p_{sr} s-spaces; therefore in R_t there are $p_{sr}p_{rt}$ s-spaces, each being counted (r, s, t) times. (This assumes that p_{sr} is the same for all the r-spaces, and (r, s, t) is the same for all the s-spaces).

Therefore $$p_{st}\,(r, s, t)=p_{sr}p_{rt}$$

and $$(r, s, t)=\frac{p_{sr}p_{rt}}{p_{st}} \qquad (1)$$

Also, through an R_s there are p_{rs} r-spaces; through each there pass p_{tr} t-spaces, therefore through R_s there pass $p_{tr}p_{rs}$ t-spaces, each being counted (r, s, t) times. (With similar assumptions).

Therefore $$p_{ts}(r, s, t) = p_{rs} p_{tr},$$

and $$(r, s, t) = \frac{p_{rs} p_{tr}}{p_{ts}} \qquad (2)$$

From (1) and (2)

$$p_{rt} p_{st} p_{tr} = p_{rt} p_{ts} p_{sr} \qquad (3)$$

(3) is proved under the limitation $t > r > s$, but the symmetry of the result shows that it is independent of this assumption.

4. *To express p_{0t} in terms of p_{01}.*

Take an R_{t-1} and a point 0 outside it; then R_{t-1} and 0 determine an R_t (A). Join 0 to each of the $p_{0,t-1}$ points of R_{t-1} (A). On each of these lines there are p_{01} points (C), and we now have *all* the points in R_t, for if there were any other point 0′, 00′ cuts R_{t-1} in a point which has already been chosen (B)

Therefore $$p_{0t} = p_{0,t-1}(p_{01} - 1) + 1 \qquad (4)$$

(4) is a reduction formula, following from (A), (B) and (C) alone, by which p_{0t} may be expressed in terms of p_{01}. p_{0t} is therefore constant.

Let $p_{01} = p + 1$, then from (4)

$$p_{0t} = p \cdot p_{0,t-1} + 1.$$

Now $$p_{02} = p_{01}(p_{01} - 1) + 1.$$

$$= p^2 + p + 1 = \frac{p^3 - 1}{p - 1}.$$

Hence assuming $$p_{0,t-1} = \frac{p^t - 1}{p - 1}$$

we get $$p_{0t} = \frac{p^{t+1} - p}{p - 1} + 1 = \frac{p^{t+1} - 1}{p - 1} \qquad (5)$$

From this it follows that

$$p_{0t} - p_{0s} = \frac{p^{t+1} - p^{s+1}}{p - 1}.$$

We have now to express p_{rt} and (r, s, t) in terms of p.

5. *To express p_{rt} $(r < t)$ in terms of p.*

An r-space is determined by $r + 1$ points. Let us choose $r + 1$ determining points in an R_t. The first point 0_1 can be chosen in p_{0t} ways, the second 0_2 in $p_{0t} - 1$ ways. A third, 0_3, not in a line with these, can be chosen in $p_{0t} - p_{01}$ ways; a fourth, not in a plane

with 0_1, 0_2, 0_3 in $p_{0t} - p_{02}$ ways, and so on. The number of ways of choosing the $r+1$ points is then

$$p_{0t}(p_{0t}-1)(p_{0t}-p_{01})(p_{0t}-p_{02})\ldots(p_{0t}-p_{0,r-1})/(r+1)!$$

Similarly, in this r-space we may choose $r+1$ determining points in

$$p_{0r}(p_{0r}-1)(p_{0r}-p_{01})(p_{0r}-p_{02})\ldots(p_{0r}-p_{0,r-1})/(r+1)!$$

ways. Hence

$$\begin{aligned} p_{rt} &= \frac{p_{0t}(p_{0t}-1)(p_{0t}-p_{01})\ldots(p_{0t}-p_{0,r-1})}{p_{0r}(p_{0r}-1)(p_{0r}-p_{01})\ldots(p_{0r}-p_{0,r-1})} \\ &= \frac{(p^{t+1}-1)(p^{t+1}-p)(p^{t+1}-p^2)\ldots(p^{t+1}-p^r)}{(p^{r+1}-1)(p^{r+1}-p)(p^{r+1}-p^2)\ldots(p^{r+1}-p^r)} \\ &= \frac{(p^{t+1}-1)(p^t-1)\ldots(p^{t-r+1}-1)}{(p^{r+1}-1)(p^r-1)\ldots(p-1)} \\ &= \frac{\Pi(p^{t+1}-1)}{\Pi(p^{r+1}-1)(p^{t-r}-1)}. \qquad (6) \end{aligned}$$

Hence $p_{r-1,t} = p_{t-r,t}$, a reciprocal relation between the number of $(r-1)$-spaces and the number of $(t-r)$-spaces in a t-space.

6. *To express (r, s, t) in terms of p.*

In R_t take an R_s. An R_r contained in R_t and passing through R_s requires $r+1$ points to determine it; $s+1$ of these are in the R_s. We can choose a first point, 0_1, outside R_s and in R_t in $p_{0t} - p_{0s}$ ways; a second, 0_2, not in the $(s+1)$-space determined by R_s and 0_1, in $p_{0t} - p_{0,s+1}$ ways, and so on. The number of ways of choosing the $r-s$ additional points is therefore

$$(p_{0t}-p_{0s})(p_{0t}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})/(r-s)!$$

Similarly in the R_r we can choose the $r-s$ additional points which are required to determine it in

$$(p_{0r}-p_{0s})(p_{0r}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})/(r-s)!$$

ways. Hence

$$\begin{aligned} (r, s, t) &= \frac{(p_{0t}-p_{0s})(p_{0t}-p_{0,s+1})\ldots(p_{0t}-p_{0,r-1})}{(p_{0r}-p_{0s})(p_{0r}-p_{0,s+1})\ldots(p_{0r}-p_{0,r-1})} \\ &= \frac{(p^{t-s}-1)(p^{t-s-1}-1)\ldots(p^{t-r+1}-1)}{(p^{r-s}-1)(p^{r-s-1}-1)\ldots(p-1)} \\ &= \frac{\Pi(p^{t-s}-1)}{\Pi(p^{r-s}-1)(p^{t-r}-1)} \qquad (7) \end{aligned}$$

Hence $(r-1,\ s-1,\ t)=p_{t-r,\,t-s}$, a reciprocal relation between the number of $(r-1)$-spaces through an $(s-1)$-space and the number of $(t-r)$-spaces in a $(t-s)$-space in R_t.

Putting $t=n$ in (7) we find for $r>s$

$$p_{rs}=\frac{\Pi(p^{n-s}-1)}{\Pi(p^{r-s}-1)(p^{n-r}-1)} \quad . \quad . \quad . \quad . \quad (8)$$

7. All the numbers in the scheme have now been expressed in terms of p, using only the assumptions (A), (B), (C), hence they are all constant and determinate when p is given. We notice also that the configuration is reciprocal since

$$p_{r-1,\,t}=p_{t-r,\,t}$$

and $$(r-1,\ s-1,\ t)=p_{t-r,\,t-s}.$$

With the help of formulæ (6) and (7) the formulæ (1), (2), and (3) may now be verified. They have not been employed in the proofs of (6) and (7), and might therefore have been proved by means of (A), (B), and (C) alone.

8. The following correspondence may now be established in the case where p is a prime.

Construct an Abelian group of order p^{n+1} in which each operation is of order p. The number of subgroups of order p^{r+1} is p_{rn} in formula (6), the number of subgroups of order p^{r+1} contained in a given subgroup of order p^{t+1} is p_{rt}, and so on.*

Hence a correspondence is established between this group and the configuration of points, lines, etc., in such a way that to a point corresponds a subgroup of order p, to a line a subgroup of order p^2, and in general, to a t-space a subgroup of order p^{t+1}. For the connection with the Galois Field theory see Veblen and Bussey l.c.

II.

9. There is a problem of arrangements connected with these configurations. Confining our attention to a plane, consider the configuration with r points in each line and r^2-r+1 points altogether. Denoting the points by numbers, we can arrange the n numbers, each repeated r times, in n sets of r each, such that any pair of numbers occurs in one and only one set.

* See Burnside, "Theory of Groups," p. 59.

For $r=4$, $n=13$ a possible arrangement is

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

where each number occurs once in each row. We also observe that each complete row is obtained from the first by a cyclic permutation.

10. Let us assume the possibility of arranging the n numbers in n sets of r each, *i.e.*, in n r-ads, and investigate the nature of the arrangement when each row contains all the numbers. The first arrangement is possible whenever there is a finite projective desarguesian geometry in the plane with n points, and this happens whenever p or $r-1$ is a prime or a power of a prime. Let us assume therefore that the n numbers, each repeated r times, are disposed in r rows and n columns in such a way that each number occurs in each row and every pair of numbers occurs in one and only one column.

Let the substitutions by which the 2nd, 3rd, ..., rth rows are obtained from the first be denoted by (12), (13), ..., $(1r)$, and consider the r columns in which a specified number p occurs. When p is in the first row the other numbers in the same column are

$$p(12),\ p(13),\ \ldots,\ p(1r);$$

when p is in the mth row the other numbers are

$$p(1m)^{-1},\ p(1m)^{-1}(12),\ \ldots,\ p(1m)^{-1}(1r).$$

These numbers, for $m=2, 3, \ldots, r$, must be all different except p itself which occurs in each set.

Hence we get $r(r-1)+1=n$ different operations

$$1,\ (12),\ (13),\ \ldots,\ (1r),\ (12)^{-1},\ (12)^{-1}(13),\ \ldots,\ \ldots,\ (1r)^{-1}(1,\ r-1),$$

or, denoting $(1p)^{-1}(1q)$ by (pq), we have the n distinct operations

$$(pq) \qquad (p,\ q=1,\ 2,\ \ldots,\ r)$$

where $(pp)=1$ and $(pq)(qs)=(ps)$.

Starting with any number p, it is transformed by these substitutions into the n different numbers of the scheme. Let S, T be any two substitutions of the set, then corresponding to p in the first row we have $p\mathrm{T}$ in the tth row, say, so that corresponding to $p\mathrm{S}$ in the first row we have $p\mathrm{ST}$ in the tth row. Therefore ST is a substitution of the set. Hence these operations form a group of order n.

Now each operation, except identity, changes all the symbols, and each changes a given symbol p into a different symbol, for if S and T both change p into q, then ST^{-1} leaves p unaltered, therefore $ST^{-1}=1$, or $S=T$. Again, since all the powers of S belong to the group, they must all change all the symbols, except that power which is the identical operation; hence each substitution must be regular.

Suppose now $n=ab$ and

$$S=(p_1 p_{a+1} p_{2a+1} \cdots p_{(b-1)a+1})(p_2 p_{a+2} \cdots p_{(b-1)a+2}) \cdots$$

then S is the ath power of

$$T=(p_1 p_2 \cdots p_a p_{a+1} p_{a+2} \cdots p_{2a} p_{2a+1} \cdots)$$

and the group is therefore the cyclical group generated by the single operation T of order n.

11. Now take any operation of order n of the group and denote it by 1 and its 2nd, 3rd, ... powers by 2, 3, ..., its nth power, which is identity, being denoted by 0. There is one set among the sets

$$(1q),\ (2q),\ \ldots,\ (rq) \qquad (q=1,\ 2,\ \ldots,\ r)$$

in which this operation occurs. Taking that set, denoted by the numbers, as the first column, the whole arrangement can be written down by writing the numbers in order in each row. Thus for $n=21$ a possible group makes (14) of order 21 and the powers of (14) are

0	1	2	3	4	5	6	7	8	9	10
1	(14)	(25)	(31)	(34)	(42)	(12)	(45)	(15)	(32)	(53)

11	12	13	14	15	16	17	18	19	20
(35)	(23)	(51)	(54)	(21)	(24)	(43)	(13)	(52)	(41)

The first column is then 0 1 6 8 18 so that we have the arrangement

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7
18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

12. There is in general considerable latitude in forming the group whose operations satisfy the given conditions. But if one arrangement has been obtained others may at once be obtained from it.

Any arrangement is completely determined when one column is given. Let the first column, expressed in terms of the operations of the group, be $$1 P_1 P_2 \ldots P_{r-1}$$
then forming the other columns which contain 1, the top row will be
$$1 P_1^{-1} P_2^{-1} \ldots P_{r-1}^{-1}$$
and if this is taken as the first column we get a new arrangement.* This gives then $2r$ different columns containing 1, r belonging to each arrangement. Again, if a is prime to n
$$1 P_1^{a} P_2^{a} \ldots P_{r-1}^{a}$$
gives another arrangement, for if S and T are any two numbers of the first arrangement S^a and T^a are distinct. If a is not prime to n, S^a may be equal to T^a without S being equal to T. If we take for a all the numbers less than n and prime to it, the resulting columns are in general all different. In particular if n be a prime, and
$$1 P P^{a_1} P^{a_2} \ldots P^{a_{r-2}}$$
is a column such that the columns formed by taking the powers of the elements are not all distinct, the only powers which give the same column are evidently $1, a_1, a_2, \ldots, a_{r-2}$; hence these must be the $r-1$ numbers which appertain (mod n) to $r-1$ and its factors. If $r-1$ is even, one of these numbers is $n-1$ since $(n-1)^2 \equiv 1 (\text{mod } n)$, and we have seen above that 1, P and P^{n-1} cannot belong to the same column, so that in this case all the columns will be different. For $r=4$, $n=13$ the numbers a are 1, 3, 9, since $27 \equiv 1 (\text{mod } 13)$ and 0, 1, 3, 9 is a possible column.

13. When n is prime and r is odd it is easy to find at least a lower limit for the number of distinct arrangements. We get first $2r$ distinct columns; then by taking powers we get from any one column $n-1 = r(r-1)$ distinct columns, so that $k \,.\, 2r = l \,.\, r(r-1)$. The least value for l is 1 and the least value for k is $\frac{1}{2}(r-1)$, therefore the number of distinct arrangements is $r-1$ or a multiple of this. If r is even the set of numbers a can be found. If these form a possible column the number of distinct columns is
$$r + (l-1) \,.\, r(r-1) = 2kr \quad \text{or} \quad 2k = 1 + (l-1)(r-1).$$

---

* See, however, the end of § 14.

The least value for l is 2 and the least value for k is $\frac{1}{2}r$. If, however, the numbers do not form a possible column, the number of distinct columns is $l.r(r-1)=2kr$ so that $l=2$ and $k=(r-1)$. $2k$ or at least a multiple of $2k$ will be the number of distinct arrangements.

14. The following are the arrangements which I have obtained for $n=3$, 7, 13, 21, 31. Each square represents the columns which contain 0, and can be read horizontally and vertically.

$n = 3$

0	2
1	0

$n = 7$

0	6	4
1	0	5
3	2	0

$n = 13$

0	12	9	7
1	0	10	8
4	3	0	11
6	5	2	0

0	12	10	4
1	0	11	5
3	2	0	7
9	8	6	0

$n = 21$

0	20	17	7	5
1	0	18	8	6
4	3	0	11	9
14	13	10	0	19
16	15	12	2	0

$n = 31$

0	30	28	23	19	13
1	0	29	24	20	14
3	2	0	26	22	16
8	7	5	0	27	21
12	11	9	4	0	25
18	17	15	10	6	0

0	30	28	21	17	5
1	0	29	22	18	6
3	2	0	24	20	8
10	9	7	0	27	15
14	13	11	4	0	19
26	25	23	16	12	0

0	30	27	25	18	10
1	0	28	26	19	11
4	3	0	29	22	14
6	5	2	0	24	16
13	12	9	7	0	23
21	20	17	15	8	0

0	30	27	21	19	14
1	0	28	22	20	15
4	3	0	25	23	18
10	9	6	0	29	24
12	11	8	2	0	26
17	16	13	7	5	0

0	30	23	20	18	14
1	0	24	21	19	15
8	7	0	28	26	22
11	10	3	0	29	25
13	12	5	2	0	27
17	16	9	6	4	0

The two arrangements which are obtained by reading the squares horizontally and vertically are really identical, differing only in notation. The symbols, written in cyclical order, being $0, 1, 2, 3, \ldots, n-1$, the substitution

$$(0)(1, n-1)(2, n-2) \ldots$$

simply reverses the cyclical order.

15. This problem is only one of a class of tactical problems connected with these configurations. E. H. Moore* has given a great many results relating to these arrangements. In his notation $S[k, l, m]$ represents a k-adic system in m letters of index 1 such that every l-ad of the system is incident with one and only one of the k-ads. The systems here considered have the index $l=2$ and are $S[r, 2, r^2-r+1]$. The general type of a tactical system to be considered here is the "finite geometry system," which may be denoted by FGS$[p_{lr}, *, p_{lt}]$ or FGS$p\,[(l, r, t), (t>r>l)$, *i.e.*, a p_{lr}-adic system in p_{lt} letters, with a reciprocal system FGS$[(l, r, t), *, p_{lt}]$ or FGSc$[l, r, t]\,(t>l>r)$, *i.e.*, an (l, r, t)-adic system in p_{lt} letters, where each p_{lr}-ad belonging to an FGS$p\,[(l, r, t)$ forms also an element in the FGS$p\,(r, s, t)$ and in the FGSc$[r, s, t]$ and each (l, r, t)-ad belonging to an FGSc$[l, r, t]$ forms also an element in the FGSc$[rst]$ and in the FGS$p\,[r, s, t]$. It may be further defined inductively thus. Every k-ad which is incident with a p_{lv}-ad belonging to an FGS$p\,[l, v, t]$ and not with a $p_{l,v-1}$-ad belonging to an FGS$p\,[l, v-1, t]$ is incident with (r, v, t) of the p_{lr}-ads; and, moreover, every k-ad of the p_{lr}-ads which is incident with an (r, v, t)-ad belonging to an FGSc$[r, v, t]$ and not with an $(r, v+1, t)$-ad belonging to an FGSc$[r, v+1, t]$ have in common a unique p_{lv}-ad.

The following is an FGS$p\,[0, 2, 3]$ with 15 $p_{02}\,(=7)$-ads:—

1	1	1	1	1	1	1	2	2	2	2	3	3	3	3
2	2	2	4	4	6	6	4	4	5	5	4	4	5	5
3	3	3	5	5	7	7	6	6	7	7	7	7	6	6
4	8	12	8	10	8	10	8	9	8	9	8	9	8	9
5	9	13	9	11	9	11	10	11	10	11	11	10	11	10
6	10	14	12	14	14	12	12	13	13	12	12	13	13	12
7	11	15	13	15	15	13	14	15	15	14	15	14	14	15

* "Tactical Memoranda," *Amer. J.*, XVIII. (1896), pp. 264–303.

where each of the 35 triads S [3, 2, 15] or FGSp [0, 1, 3].

1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3
2	4	6	8	10	12	14	4	5	8	9	12	13	4	5	8	9	12	13
3	5	7	9	11	13	15	6	7	10	11	14	15	7	6	11	10	15	14

4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7
8	9	10	11	8	9	10	11	8	9	10	11	8	9	10	11
12	13	14	15	13	12	15	14	14	15	12	13	15	14	13	12

is incident with three of the 7-ads, each of the remaining 420 triads being incident with one and only one 7-ad. Also any pair of 7-ads have a unique triad in common, each pair of elements is incident with one and only one triad, and each element is incident with 7 of the 7-ads and with 7 of the triads.

Pythagoras's Theorem.*

By M. EDOUARD COLLIGNON.

The square described on the hypotenuse of a right-angled triangle is equal to the squares described on the other two sides.

About half a hundred proofs of this theorem have been given, but few of them have been "ocular," that is, few have shown how the two smaller squares may be decomposed so as to fit into the largest square. One of the most elegant of the ocular proofs is that of Henry Perigal, and was discovered about 1830. A demonstration of its correctness is not difficult to obtain, but the following demonstration is believed to be new. It depends somewhat on algebra, and presupposes a simple lemma.

Perigal's Construction.

FIGURE 23.

Triangle ABC is right-angled.
BCED is the square on the hypotenuse, ACKH and ABFG are the squares on the other sides.

Find O the centre of the square ABFG, which may be done by drawing the two diagonals (not shown in the figure), and through it draw two straight lines, one of which is parallel to BC, and the other perpendicular to BC. The square ABFG is then divided into four quadrilaterals equal in every respect.

Through the mid points of the sides of the square BCED draw parallels to AB and AC as in the figure.

The parts numbered 1, 2, 3, 4, 5 will be found to coincide with 1′, 2′, 3′, 4′, 5′.

Lemma. The square described on a diagonal of a square is double of the square described on a side.

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* This is the 47th proposition of the 1st book of Euclid's *Elements*, and is called in France *le pont aux ânes*, the *asses' bridge*. This name, in English-speaking countries, is bestowed on the 5th proposition of the 1st book of Euclid's *Elements*.

FIGURE 24.

This is easily seen from an inspection of Figure 24, and from what is known about congruent triangles.

Triangles 1, 3 may be slid (that is, without moving them out of the plane) vertically into coincidence with 1′, 3′; and triangles 2, 4 may be slid horizontally into coincidence with 2′, 4′.

Demonstration of the Theorem.

FIGURE 25.

From the properties of parallelograms

$$MM' = CB \text{ and } NN' = MM'.$$

Denote the sides BC, CA, AB by a, b, c; and calculate the segments AM, AN, ... determined by the vertices of the inscribed square MNM′N′ on the sides of the square ABFG.

From the construction of the figure, we have

$$AM = BN \qquad OI = \frac{c}{2} \qquad ON = OM = \frac{a}{2}.$$

By the lemma, the side of the inscribed square,

$$MN = \frac{a}{2}\sqrt{2} = \frac{a}{\sqrt{2}}.$$

Hence the area of the inscribed square is $\frac{a^2}{2}$ and the area of the four equal triangles AMN, ... is the difference $c^2 - \frac{a^2}{2}$.

Each triangle is therefore the quarter of this difference, that is,

$$\tfrac{1}{4}\left(c^2 - \frac{a^2}{2}\right).$$

But the area of each of the triangles

$$= \tfrac{1}{2}AM \times AN.$$

Hence
$$AM \times AN = AN \times NB = \frac{c^2}{2} - \frac{a^2}{4},$$

and the product of the segments AN, NB is known.

Now the sum of the segments AN, NB is also known, since it is equal to AB or c.

The two segments are therefore the roots of an equation of the second degree

$$(1) \qquad t^2 - ct + \frac{c^2}{2} - \frac{a^2}{4} = 0\,;$$

whence, after simplification,

$$(2) \qquad t = \frac{c}{2} \pm \tfrac{1}{2}\sqrt{a^2 - c^2}\,.$$

The difference of the roots is therefore equal to the radical

$$\sqrt{a^2 - c^2}\,;$$

and if, on the segment AN, we cut off AP equal to BN, this difference is PN.

The mid point of PN is I, and consequently

$$\mathrm{IN} = \tfrac{1}{2}\sqrt{a^2 - c^2}\,.$$

Since triangle OIN has its sides respectively perpendicular to those of the given triangle BAC, these two triangles are similar; therefore

$$(3) \qquad \frac{\mathrm{IN}}{\mathrm{AC}} = \frac{\mathrm{ON}}{\mathrm{BC}} = \frac{\mathrm{OI}}{\mathrm{BA}} = \tfrac{1}{2}\,.$$

Hence $\qquad \mathrm{IN} = \dfrac{b}{2}$ and $\mathrm{PN} = b$.

Finally, we have the relation

$$(4) \qquad b = \sqrt{a^2 - c^2}\,,$$

that is, $\qquad b^2 + c^2 = a^2\,,$

which proves the theorem.

Post-scriptum.

On peut justifier le théorème sur l'un quelconque des triangles AMN qu'on forme pour la démonstration de Henry Perigal.

On connaît les trois côtés de ce triangle : $\mathrm{MN} = \dfrac{a}{\sqrt{2}}$, a désignant l'hypothénuse CB ; AM et AN sont les racines d'une équation du second degré

$$t^2 - ct + \frac{c^2}{2} - \frac{a^2}{4} = 0.$$

Sans résoudre l'équation, appelons t' et t'' les racines ; nous avons à former $t'^2 + t''^2$ et vérifier que cette somme reproduit $\frac{a^2}{2}$.

Or, la somme des carrés des racines d'une équation algébrique quelconque

$$x^m + px^{m-1} + qx^{m-2} + \ldots = 0$$

s'exprime par la fonction $p^2 - 2q$, car $-p$ est la somme des racines, et q la somme de leurs produits deux à deux. Ici on a

$$p = -c \qquad q = \frac{c^2}{2} - \frac{a^2}{4}$$

et

$$p^2 - 2q = c^2 - \left(c^2 - \frac{a^2}{2}\right) = \frac{a^2}{2}$$

ce qui justifie le résultat annoncé.

On a Problem in Rigid Dynamics.

By G. M. K. Leggett.

Seventh Meeting, 14th June 1907.

J. Archibald, Esq., M.A., President, in the Chair.

Herbert Spencer and Mathematics.

By John Sturgeon Mackay, M.A., LL.D.

About forty years ago, when Spencer was rising into philosophic fame, it used often to be said by his admirers that he was an accomplished mathematician. This statement was accepted without demur, though it was known that he had not measured himself against rivals of his own age, or, what is more important, had not produced anything new in this old science.

Since his *Autobiography** has been published, an estimate can be formed from his own statements of what were his acquirements in this subject, and what were his contributions to it.

If the estimate which follows seems severe, it must be remembered that Spencer was an unsparing critic of others. Of his own character he has said :—

"No one will deny that I am much given to criticism. Along with exposition of my own views there has always gone a pointing out of defects in the views of others. And if this is a trait in my writing, still more is it a trait in my conversation. The tendency to fault-finding is dominant—disagreeably dominant." (II. 438)

He has also said :

"It has been remarked that I have an unusual faculty of exposition—set forth my data and reasonings and conclusions with a clearness and coherence not common." (II. 437)

We shall have examples of this clearness of exposition later on.

In summing up the results of his education till the age of

* This consists of two huge volumes, the first containing 556, and the second 542 pages. The preface by himself is dated 27th April 1894, when he was 74 years of age ; he was born on 27th April 1820. The publication of the *Autobiography* by his trustees took place in 1904.

To save repetition of *Autobiography* when the book is referred to, the volume and the page have alone been noted.

thirteen, he says, among other instructive remarks about what he did not know, that

"I had merely the ordinary knowledge of arithmetic; and beyond that no knowledge of mathematics."

The ordinary knowledge of arithmetic would be an acquaintance with the simple and compound rules. Mathematics at that time meant Geometry and Algebra, more frequently Geometry alone.

Towards the end of June 1833 he was taken by his parents to visit his uncle Thomas at Hinton Charterhouse, near Bath, and was left there to be educated.

"I had supposed I was about to spend a month's summer holidays; but I was taken by my uncle one morning and set down to the first proposition in Euclid. Having no love of school or of books, this caused in me great disgust. However there was no remedy, and I took to the work tolerably well: my faculty lying more in that direction than in the directions of most subjects I had dealt with previously. This was significantly shown before the end of a fortnight; when I had reached perhaps the middle of the first book. Having repeated a demonstration after the prescribed manner up to a certain point, I diverged from it; and when my uncle interrupted me, telling me I was wrong, I asked him to wait a moment, and then finished the demonstration in my own way; the substituted reasoning being recognised by him as valid." (I. 93)

At Hinton he learned Euclid and Latin in the morning, and in the evening some Alegbra. Towards the end of October he says "there is mention of demonstrations made by myself of propositions in the fourth book of Euclid: not, however, approved by my uncle."

In a letter to his father dated January 28, 1834, there occurs the passage:

"I forgot to tell you in my last letter that I had made some problems in Algebra with which my uncle was much pleased, and as I want something to fill up I will tell you them all. My uncle was most pleased with the 5th of these which he thought was very original."

"Correspondence shows that in March I was learning... Trigonometry. With Trigonometry I speak as being delighted: sending my father some solutions of trigonometrical questions."

"Euclid was gone through again at this time; and mention is made of the fact that I was able to repeat some of the propositions

without the figures: not, as might be supposed, by rote-learning, but by the process of mentally picturing the figures and their letters, and carrying on the demonstrations from the mental pictures." (I. 104–105)

What is meant by "Euclid" is very vague, and the remark that he was able to repeat some of the propositions without the figures is naive. Why it might be supposed that he did it by rote-learning is significant of some of the geometrical teaching which at that time prevailed.

Spencer does not seem ever to have been aware that, before his time, the constructions and demonstrations of several of Euclid's books "in general terms" had been published, the diagrams being left to the readers' imagination, and no letters being required.

"Refore the end of May [1835] I had been through the eleventh book of Euclid, and also through 'Lectures on Mechanics'—either Wood's Mechanics, a text-book in my uncle's college days, which I certainly went through at some time, or else the Cambridge Lectures which he had written down, and which we studied from his MS."

"In a letter to my father dated July 28, I apologise for breaking off because 'I have to learn a quantity of Newton to keep up with the others this morning'; and there occurs the sentence—'But I am very proud of having got into Newton.' Reference to the MS. book, which I still possess, shows that I did not go very far." (I. 110)

"That which remained with me best was the mathematical knowledge I acquired; for though the details of this slipped, I readily renewed them. Thus in May 1836 I describe myself in a letter as going through six books of Euclid in a week and a half." (I. 115.)

In the appendices to each of the volumes of his *Autobiography* Spencer reproduces some of his articles to periodicals and some memoranda he made when he was a young man.

Part of those which refer to mathematical matters are here extracted.

"It was either during the autumn of 1836 or during that of 1837 that I hit upon a remarkable property of the circle, not, so far as I have been able to learn, previously discovered . . . . I did not then attempt a proof. This was not supplied until some two years later." (I. 119)

"When seventeen I hit on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but . . . responding to a spur from my father, I made a demonstration of it; and now that it had reached this developed form, it was published in *The Civil Engineer and Architect's Journal* for July 1840. . . . . I did not know, at the time, that this theorem belongs to that division of mathematics at one time included under the name *Descriptive Geometry*, but known in more recent days as *The Geometry of Position*—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seemingly so utterly incomprehensible. The property of a circle which is enunciated in my own theorem has nothing like so marvellous an aspect, but is nevertheless sufficiently remarkable." (I. 164)

The fact that Spencer allowed his "remarkable property of the circle" to remain with him "in the form of an empirical truth" (!) implies (Spencer is very fond of implications) that he was at this time somewhat devoid either of geometrical ardour or of geometrical skill.

But before going farther it may be well to enunciate what Spencer calls "his own theorem."

"Geometrical Theorem

"Sir

"I believe that the following curious property of a circle has not hitherto been noticed; or if it has, I am not aware of its existence in any of our works on Geometry."

[The reader is requested to make the figure.]

"Let ABCDE [read clockwise] be a circle of which ACD is any given segment: Let any number of triangles ABD, ACD, etc., be drawn in this segment, and let circles be inscribed in these triangles; their centres F, G, etc., are in the arc of a circle, whose centre is at E, the middle of the arc of the opposite segment AED."

The theorem is undoubtedly true, but Spencer's diagram is

unnecessarily complicated, and his demonstration rather verbose. The property however which Spencer signalises had been noticed long before his time.

This is how the theorem arises.

Let a circle be circumscribed about a triangle ABC (call its centre O), and another circle be inscribed in ABC (call its centre I); to find the expression for the distance OI in terms of the radii of the two circles.

The first mathematician to find this expression was William Chapple * in 1746, but in his demonstration the property discovered by Spencer does not appear. It does appear however in the solution given by John Turner in 1748 in *The Mathematician*, p. 311, to the problem :

One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically.

Spencer's property is also given by Nicolas Fuss in 1794. See the 10th volume of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* (Petropoli, 1797).

In the third † part of "The Elements of Plane Geometry" by the Rev. J. Luby, p. 57, among the exercises on Loci occur

Given the base and vertical angle of a triangle, required the loci

(*a*) of the centre of the inscribed circle

(*b*) of the centre of the circle that touches the base and the two sides produced

(*c*) of the centre of the circle that touches one side and the productions of the base and other side.

Exercise (*a*) is Spencer's theorem under another guise, and (*b*) and (*c*) are extensions of it.

The remarkable property of the circle is, as far as Spencer is concerned, the fact that he remarked it. It is, if one wished to describe it more accurately, a property of the triangle, and has nothing whatever to do with "Descriptive Geometry," which Spencer confounds with "Geometry of Position" (*Die Geometrie der*

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* *Miscellanea Curiosa Mathematica*, p. 117–124.

† The title-page is not dated, but the preface to the first part ends with Trinity College [Dublin] Sept. 7, 1833.

Lage of the Germans) or what is now called "Projective Geometry." It is a metrical property, like those expounded by Euclid.

The familiar truth which Spencer never contemplates without being struck by its beauty, etc., is expressed by him in terms of singular inexactitude.

"If to three unequal circles" [They needn't be unequal] "anywhere placed" [For example, in different planes?] "three pairs of tangents be drawn" [Millions of pairs of tangents may be drawn to three circles. What is implied and should have been stated is that the tangents must be common to every pair of the three circles. Furthermore, Spencer does not seem to have known that six pairs of common tangents can be drawn to three circles, taken two by two, and that the points of intersection of these six pairs give rise to four straight lines. From the phrase "anywhere placed" Spencer does not seem to have imagined any position of the three circles in which pairs of common tangents were impossible; for example when the first circle is inside the second, the second inside the third, and there is no mutual contact.]

The feelings of wonder and awe excited by "the fact that apparently unrelated circles should in every case be held together by this plexus of relations seemingly so utterly incomprehensible" probably arose from Spencer's unfamiliarity with any geometrical truths outside the first six books of Euclid. The circles are not "apparently unrelated." To any fairly well read geometer the following relations are evident at a glance:

(1) The circles are in the same plane.

(2) Being circles they are similar figures.

(3) Every pair of them may be regarded as similarly situated, that is, as having an external centre of similitude.

(4) Every pair of them may be regarded as oppositely situated, that is, as having an internal centre of similitude.

The application of a few geometrical theorems to the relations just stated soon removes any utter incomprehensibility.

The following extract from the last book Spencer published, *Facts and Comments* (1902) pp. 203–4, is not easy to characterise.

"In youth we pass without surprise the geometrical truths set down in our Euclids. It suffices to learn that in a right-angled triangle the square of the hypotenuse is equal to the sum of the

squares of the other two sides: it is demonstrable, and that is enough. Concerning the multitudes of remarkable relations among lines and among spaces very few ever ask—Why are they so? Perhaps the question may in later years be raised, as it has been in myself, by some of the more conspicuously marvellous truths now grouped under the title of 'the Geometry of Position.' Many of these are so astounding that but for the presence of ocular proof they would be incredible; and by their marvellousness, as well as by their beauty, they serve, in some minds at least, to raise the unanswerable question—How come there to exist among the parts of this seemingly-structureless vacancy we call Space, these strange relations? How does it happen that the blank form of things presents us with truths as incomprehensible as do the things it contains?"

The phrases "very few ever ask" and "they serve in some minds at least" betray Spencer's consciousness of his superiority to other people; and the remark that "but for the presence of ocular proof" certain marvellous truths "would be incredible" indicates a very slight acquaintance with the properties of geometrical figures, as well as a very humble standard by which to judge of mathematical truths.

What truths incomprehensible or not are presented to us by a "seemingly-structureless vacancy" or by a "blank form of things" I am unable to conceive.

In the description of his visit to America, Spencer gives us a glimpse into his knowledge of Mathematical Geography.

"While sitting on a ledge of rock facing the East, and looking over the wide country stretching away to the horizon below the Hudson, it was interesting to think that here we were in a land we had read about all our lives—interesting, and a little difficult, to think of it as some three thousand miles from the island on the other side of the Atlantic whence we had come. Not easy was it either, and indeed impossible in any true sense, to conceive the real position of this island on that vast surface which slowly curves downward beyond the horizon: the impossibility being one which I have vividly felt when gazing sea-ward at the masts of a vessel below the horizon, and trying to conceive the actual surface of the Earth, as slowly bending round till its meridians met eight thousand

miles beneath my feet: the attempt producing what may be figuratively called a kind of mental choking, from the endeavour to put into the intellectual structure a conception immensely too large for it." (II. 390)

Many pupils in a Geography class know that meridians meet at the North and the South Poles, and that any diameter of the Earth, that is, a straight line passing through its centre and terminated both ways by its surface, is approximately eight thousand miles. Spencer was never south of the Equator; hence he must have been at the North Pole to conceive even imperfectly what he "vividly felt."

Here is a glimpse into his knowledge of Mathematical Astronomy.

"When, many centuries after, Kepler discovered that the planets moved round the Sun in ellipses, and described equal areas in equal times . . . " *First Principles*, p. 103 (3rd ed. 1870)

I need not quote Kepler's three laws, but I may draw attention to the inadequacy of Spencer's statement of two of them. He omits to say what was the position of the Sun, namely, in one of the foci of the ellipses, and the description of equal areas in equal times ought to be attributed, not to the planets, but to the radii vectores of the planets, that is, to the straight lines drawn from the Sun to the planets.

Spencer's attitude towards the proposal to adopt the Metric System was one of uncompromising opposition, and in 1896 he published an ill-digested pamphlet entitled "Against the Metric System." I do not intend to discuss Spencer's arguments showing that the Metric is "a very imperfect system." The most important of these arguments have been answered time after time, and the others are puerile in the extreme. It is well known that the Metric System is not perfect, but the great difficulty is to get a better one. Here is Spencer's solution of the difficulty. I shall give it as far as possible in his own words, only remarking that the language he sometimes employs is very loose. He speaks of numeration and notation as if they were identical, and does not seem to know that while the decimal system of numeration has been adopted almost exclusively since the dawn of civilisation, or at any rate for several thousands of years, the systems of notation have varied considerably.

"We agree in condemning the existing arrangements under which our scheme of numeration and our modes of calculation based on it proceed in one way, while our various measures of length, area, capacity, weight, value proceed in other ways. Doubtless, the two methods of procedure should be unified; but how? You [addressing an opponent] assume that, as a matter of course, the measure-system should be made to agree with the numeration-system; but it may be contended that, conversely, the numeration-system should be made to agree with the measure-system—with the dominant measure-system, I mean."

If the British tables of measures be consulted it will be found that there is no dominant system. Of the tables of measures in use among the peoples of the continent of Europe before the introduction of the Metric System, Spencer says, probably because he knew, nothing.

The following quotation from memoranda which Spencer made more than 50 years before he issued his pamphlet will show what he means by the dominant measure-system.

"The fact that 12 has been so generally chosen [by whom?] as a convenient number for enumeration of weights and measures, is presumptive proof that it must have many advantages. We have 12 oz. = 1 pound in Troy weight and Apothecaries weight, 12 pence = 1 shilling, 12 months in the year, 12 signs to the Zodiac, 12 lines to the inch, 12 inches to the foot, 12 sacks one last, and 12 digits. Of multiples of 12 we have 24 grains one pennyweight, 24 sheets one quire, 24 hours one day, 60 minutes one hour, 360 degrees to the circle." (I. 531)

In reference to the preceding it may be asked, Who ever uses the Troy or Apothecaries' pound, or talks of the twelfth of an inch as a line? I have never heard any one speak of a last as containing 12 sacks (a sack may be almost any size), and the 12 digits I confess to be beyond me. All this ludicrous parade of 12's is intended to help in showing that 12 is a convenient number for a base of numeration. But to continue:

"During previous years [that is, before 1842] I had often regretted the progress of the decimal system of numeration; the universal adoption of which is by many thought so desirable. That it has sundry conveniences is beyond question; but it has also sundry inconveniences, and the annoyance I felt was due to a

consciousness that all the advantages of the decimal system might be obtained along with all the advantages of the duodecimal system, if the basis of our notation were changed—if instead of having 10 for its basis, it had 12 for its basis: two new digits being introduced to replace 10 and 11, and 12 times 12 being the hundred. Most people are so little able to emancipate themselves from the conceptions which education has established in them, that they cannot understand that the use of 10 as a basis is due solely to the fact that we have five fingers on each hand and five toes on each foot. If mankind had had six instead of five, there never would have been any difficulty."

As regards the statement "most people . . . cannot understand that the use of 10 as a basis is due solely to the fact that we have five fingers on each hand," it may be remarked that nearly all pupils above the most elementary stage in all the schools of the world understand this. The statement that if mankind had had six fingers instead of five no difficulty would ever arise in calculation shows that Spencer's acquaintance with the properties of numbers was not very profound. He evidently did not know that whatever number be taken as the base of the numeration system certain difficulties would arise, and he evidently did not know that if 10 were displaced, other bases such as 6, 8, 16, 24, 60, . . . might be put forward as successors.

As regards his proposal to change, from 10 to 12, the basis of the system of numeration which prevails through the world, Spencer says

"I fully recognise the difficulties that stand in the way of making such changes as those indicated—difficulties greater than those implied by the changes which the adoption of the metric system involves. The two have in common to overcome the resistance to altering our tables of weights, measures, and values; and they both have the inconvenience that all distances, quantities, and values, named in records of the past, must be differently expressed. but there would be further obstacles in the way of a 12-notation system. To prevent confusion different names and different symbols would be needed for the digits, and to acquire familiarity with these, and with the resulting multiplication-table would, of course, be troublesome: perhaps not more troublesome, however, than learning the present system of numeration and calculation as carried on in

another language. There would also be the serious evil that, throughout all historical statements, the dates would have to be differently expressed; though this inconvenience, so long as it lasted, would be without difficulty met by enclosing in parenthesis in each case the equivalent number in the old notation. But, admitting all this, it may still be reasonably held that it would be a great misfortune were there established for all peoples and for all time a very imperfect system, when with a little more trouble a perfect system might be established."

Nearly every sentence in the preceding paragraph calls for comment, but it would be tiresome to go into complete detail. We know what the metric system of measures is, and the nomenclature by which the various denominations in any table are connected together; but no man knows what Spencer's system (to call it so) of measures would be. It couldn't be duodecimal, and have a nomenclature that would be pronounceable.

Spencer says that "to prevent confusion different names and different symbols would be needed for the digits." The digits which now exist are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and these are common to nearly all civilised peoples. Why should their forms be changed, and why should the names given to them in the various languages be changed? What body of men could devise a better set of symbols, and what set of symbols would be agreed upon by all nations? The fact is that if a duodecimal system of numeration could be established, two new symbols for ten and eleven would be all that is necesssary for the notation (how they could be fixed upon it is impossible to conjecture), but all the words in *every* language denoting numbers higher than twelve would have to be altered. Is it conceivable that all civilised nations would agree to make this stupendous change, aud how could the change be carried out? No government in the world could impose on its subjects such a modification of their language.

It is significant of Spencer's ignorance of any language but his own that he should put the learning of the present system of numeration and calculation as carried on in another language on a par with the learning of the names of his new "digits" and with the multiplication-table which would thence result.

Think, again, of some of the consequences of setting up a duodecimal system of numeration. Every arithmetical and algebraical

book in existence whether ancient or modern would be rendered useless, so would all the logarithmic tables in the world, and the hundreds of other tables of all sorts which save labour to the modest computer as well as to the profoundest mathematician.

Think of the vast body of statistics of every kind which every nation possesses, from the records of the observations of its scientific men down to the records of the population of the humblest villages. No one would think of converting these hundreds of millions of numbers from the decimal to the duodecimal scale, and reprinting them, and if the trouble of conversion had to be undergone each time a table was consulted the table might almost as well be non-existent.

Spencer remarks that throughout all historical statements the dates would have to be differently expressed. There does not appear to be any evidence that he had made a study of the Calendar, else he would have known what an amount of confusion and trouble has been caused all the world over (and indeed is still caused) by the change of dates. The Calendar plays a great part in the life of every one of us, and his proposal if it could be carried out would render every printed calendar in existence utterly useless.

It is abundantly clear from the *Autobiography* that Spencer's outfit of mathematical (or indeed any other) knowledge was both slender and scrappy, but it might have been thought that a man with even a small degree of insight into practical affairs would have hesitated before laying before his countrymen, in his mature age, the lucubrations of his youth, the memoranda of which had lain unused among his papers for more than half a century. A proposal so gigantic in its aims and so preposterous in its results could only have been conceived in ignorance and begotten of self-conceit.

Spencer asks, "Do I think this system will be adopted? Certainly not at present—certainly not for generations . . . . But it is, I think, not an unreasonable belief that further intellectual progress may bring the conviction that since a better system would facilitate both the thoughts and actions of men, and in so far diminish the friction of life throughout the future, the task of establishing it should be undertaken."

If any prophecy concerning human affairs in the long distant future is likely to be fulfilled, that surely is the one which predicts that mankind will never cut themselves off from the past by abandoning the decimal system of numeration.

Elementary Methods for Calculating First and Second Moments of Simple Configurations.

By R. F. MUIRHEAD, M.A., D.Sc.

1. The term *Second Moment*, which is already in frequent use, as applied to lines, areas and volumes, as well as masses, is preferable to the older term *Moment of Inertia* which properly applies only to masses.

2. The First Moment of a body, or of a volume, or of an area, or in fact of anything which can be conceived as composed of elements having magnitude and position, is defined with reference to a plane, an axis, or a point by the symbol $\Sigma(e.r)$ where e is the magnitude of an element, and r the measure of its distance from the plane, axis, or point of reference. The first of these quantities, the *planar* First Moment, is the most important, though in the case of plane figures the moment as to a line in the plane is identical with the moment as to a perpendicular plane passing through the line.

3. The Second Moment of a configuration is defined by the formula $\Sigma(e.r^2)$, and here again it may be *planar*, *axial*, or *polar*, according as r is the distance of e from a plane, an axis, or a point. In Dynamics the *Axial* Second Moment is that which has most direct importance, but it is simpler to begin with the calculation of *planar* moments, and every Axial Second Moment is the sum of the planar Second Moments with reference to two mutually perpendicular planes having the axis as their line of intersection. - - (1)

First Moments and Centres of Inertia.

4. The Centre of Inertia or Centroid of a system of elements Σe is defined as that point whose distance $\bar{x}$ from any plane of reference is given by $\bar{x}.\Sigma e = \Sigma(e.x)$.

The unique existence of such a point admits of easy proof. It is obvious, too, that for similar configurations similarly situated with reference to the plane of reference, $\bar{x}$ will be proportional to their linear dimensions.

5. Consider now a uniform straight line or thin rod AB whose mid point is C, whose length is a and whose mass is m. Taking moments with reference to the plane through A perpendicular to AB, and equating the moment of AB to the sum of the moments of AC and CB, we get, if $\bar{x}$ be the distance of the centroid of AB from A,

$$m\bar{x}=\frac{m}{2}\cdot\frac{\bar{x}}{2}+\frac{m}{2}\left(\frac{\bar{x}}{2}+\frac{a}{2}\right)$$

$$\therefore\quad \bar{x}=\frac{a}{2}.$$

Thus the centroid of a uniform straight line is its mid-point.

6. If we divide a rectangle into four similar rectangles each containing $\frac{1}{4}$ of its area, or a cuboid into eight similar cuboids each containing $\frac{1}{8}$ of its volume, and equate the moment of the whole to the sum of the moments of the parts with reference to a side of the rectangle or a face of the cuboid, we shall find the mid-point of the figure to be its centroid. And, more generally, if h be the height of any right or oblique prismatic body and $\bar{x}$ the distance of its centroid from one end, we shall find $\bar{x}=\frac{h}{2}$ by considering the prism as made up of an infinite number of thin uniform rods each of which has its centroid at distance $\frac{h}{2}$ from the end.

7. Next take a uniform triangular lamina ABC of mass m, such that the perpendicular from A on BC is of length h, and consider its first moment as to the plane through A perpendicular to the altitude h. Let D, E, F be the mid-points of the sides BC, CA, AB respectively, and let $\bar{x}$ be the distance of the centroid from the plane of reference. Then the four triangles AFE, FBD, EDC, DEF have their centroids respectively at distances

$$\frac{\bar{x}}{2},\ \frac{h}{2}+\frac{\bar{x}}{2},\ \frac{h}{2}+\frac{\bar{x}}{2},\ h-\frac{\bar{x}}{2}.$$

Using the same principle as before, we get

$$m\bar{x}=\frac{m}{4}\left\{\frac{\bar{x}}{2}+2\left(\frac{h}{2}+\frac{\bar{x}}{2}\right)+h-\frac{\bar{x}}{2}\right\}$$

$$\therefore\quad x=\frac{2h}{3}.$$

8. Consider next a uniform tetrahedron OABC, such that OA, OB, OC are mutually perpendicular, the length of OA being h.

Let D, E, F, H, K, L be the mid-points of BC, CA, AB, OA, OB, OC respectively, and P the 8th corner of the cuboid whose other 7 corners are O, H, K, L, D, E, F. Then the tetrahedron OABC together with the smaller one PDEF make up the three tetrahedra AFHE, BFKD, CDLE and the cuboid mentioned. Hence, taking moments as to the plane through A parallel to OBC, we have

$$m\bar{x}+\frac{m}{8}\left(h-\frac{\bar{x}}{2}\right)=\frac{m}{8}\left\{\frac{\bar{x}}{2}+2\left(\frac{h}{2}+\frac{\bar{x}}{2}\right)\right\}+\frac{6m}{8}\cdot\frac{3h}{4}$$

$$\therefore\quad \bar{x}=\frac{3h}{4}.$$

Second Moments and Radii of Gyration.

9. We shall apply to the same set of figures as above the same principle that the sum of the moments of the parts is equal to the moment of the whole; but in place of the theorem of the unique centroid we shall use the Theorem of Parallel Planes which is analogous to Huyghens' Theorem of Parallel Axes, viz.,

$$\Sigma mx^2=\bar{x}^2\Sigma m+\Sigma mx'^2,\ \text{where } x=\bar{x}+x'.$$

And in place of the theorem that for similar figures similarly placed, $\bar{x}$ is proportional to the linear dimensions, we have to note that for equi-dense similar figures similarly situated as to the plane of reference, Σex^2 is proportional to the 3rd, 4th or 5th power of a length according as the configuration is one-dimensional, two-dimensional, or three-dimensional.

10. Denoting by M the Second Moment of the unform rod previously described, as to a plane through the centroid parallel to the original plane of reference, we have

$$\mathrm{M}+m\left(\frac{a}{2}\right)^2=\frac{m}{2}\left\{\left(\frac{a}{4}\right)^2+\left(\frac{3a}{4}\right)^2\right\}+2\frac{\mathrm{M}}{8}$$

$$\therefore\quad \mathrm{M}=\frac{ma^2}{12}.$$

It is obvious that the same result would apply to any prismatic body referred to the plane at one end; and this, too, whether the prism is right or oblique, if we take a to denote the perpendicular height of the prism.

11. Similarly, for the triangle, taking moments as to the same plane of reference, and denoting by M its Second Moment as to a parallel plane through the centroid, we have

$$\mathrm{M} + m\left(\frac{2h}{3}\right)^2 = \frac{\mathrm{M}}{16} \times 4 + \frac{m}{4}\left\{\left(\frac{h}{3}\right)^2 + 2\left(\frac{5h}{6}\right)^2 + \left(\frac{2h}{3}\right)^2\right\}$$

$$\therefore \quad \mathrm{M} = \frac{mh^2}{18}.$$

Hence the Second Moment as to the parallel plane through A is

$$\frac{mh^2}{18} + m\left(\frac{2h}{3}\right)^2 = m\frac{h^2}{2}.$$

12. Again the corresponding equation for the tetrahedron gives

$$\mathrm{M} + m\left(\frac{3h}{4}\right)^2 + \frac{\mathrm{M}}{32} + \frac{m}{8}\left(\frac{5h}{8}\right)^2$$

$$= 3\frac{\mathrm{M}}{32} + \frac{m}{8}\left(\frac{3h}{8}\right)^2 + 2\frac{m}{8}\left(\frac{7h}{8}\right)^2 + \frac{3m}{4}\,\frac{h^2}{48} + \frac{3m}{4}\left(\frac{3h}{4}\right)^2$$

$$\therefore \quad \mathrm{M} = \frac{3mh^2}{80}.$$

Hence the Second Moment as to the plane of reference through A is equal to $\dfrac{3mh^2}{80} + m\left(\dfrac{3h}{4}\right)^2 = \dfrac{3mh^2}{5}$; and as to the plane of the base it is $\dfrac{3mh^2}{80} + m\left(\dfrac{h}{4}\right)^2 = \dfrac{mh^2}{10}$.

It is obvious that these formulæ will apply to any pyramid whatever, taken with reference to planes parallel to the base; for such a pyramid could be dissected into infinitesimal triangular pyramids with the same apex and height, and it is obvious that, with reference to such a plane, an oblique tetrahedron has the same Second Moment as a right one with equal base and height.

13. The Second Moment of a uniform circular lamina of mass m and radius r, about a perpendicular axis through its centre can now be found by supposing it dissected into infinitesimal triangles of mass e, whose vertices are at the centre. For each of these triangles the Second Moment relative to this axis is the same as that relative to a plane through this axis perpendicular to the mid line of the triangle, and is therefore $er^2/2$.

Hence the required Second Moment of the lamina is $\Sigma(er^2/2)$ and therefore $= mr^2/2$.

Using the theorem (1) we at once find the Second Moment of the disc relative to any plane through its axis to be $mr^2/4$, and this is of course the same as its Second Moment as to a diameter of the circle as axis.

14. It is clear that for a uniform solid cylinder, which may be looked on as composed of uniform thin circular discs, the Second Moment with reference to its axis is also $mr^2/2$. As it is prismatic, its Second Moment as to the plane of one end is $\frac{1}{3}mh^2$; while as to a diameter of the end, its axial Second Moment, by (1), is $\dfrac{mh^2}{3}+\dfrac{mr^2}{4}$.

15. Again, if a uniform solid sphere be dissected into infinitesimal tetrahedra each having mass e and a vertex at the centre, the planar Second Moment of any tetrahedron as to a diametral plane perpendicular to its length, and therefore its polar Second Moment as to the centre, would be $3er^2/5$.

Hence the polar Second Moment of the solid sphere as to the centre is $\Sigma(er^2/5)$ or $3mr^2/5$; from which it follows that the Second Moment as to a diametral plane is $mr^2/5$, and that as to a diameter is $2mr^2/5$.

16. By the aid of the well-known and obvious principle that a pure longitudinal strain applied to a configuration in a direction perpendicular to the plane of reference, without change of mass, will increase the Second Moment as to that plane in proportion to the square of a length in that direction, we can at once extend the results for circles and cylinders and spheres so as to apply to ellipses and elliptic cylinders and ellipsoids, so far as their principal planes and axes are concerned.

17. It is perhaps worth noting that if a body be symmetrical as to a plane, or as to some axis or point in that plane, the Second Moment of one of the two halves into which it is divided by the plane in question, taken with reference to that plane, is exactly half of the Second Moment of the whole body. In this way we can, for example, get the Second Moments of a semicircular lamina as to its

diameter, of the quadrant of a circle with reference to a radial boundary as axis, and those of the hemisphere, of the quadrant of a solid sphere, and of the octant of a solid sphere, with reference to their plane faces.

18. To find the Second Moment of a uniform solid right circular cone about its axis, consider a right circular cylinder whose length is equal to its diameter. The sphere inscribed in this cylinder is equal to the "remainder" of the cylinder when a double cone has been taken away whose bases are the ends of the cylinder. For if we take any coaxial cylindric surface and the inscribed sphere, the surface of the latter is obviously equal to that part of the cylindric surface which lies within the "remainder." Hence if the "remainder" and the sphere be divided into elementary shells by an infinite number of such surfaces, the corresponding elements will be equal, and each element of the "remainder" will have the same Second Moment as to the axis of the cylinder as the corresponding spherical element has as to the centre.

Thus the axial Second Moment of the "remainder" about its axis is equal to the polar Second Moment of the sphere as to its centre, viz., $4m \times 3r^2/5$, where m is the mass of one half of the double cone. (This result might also have been arrived at without reference to the sphere, by dividing the "remainder" into infinitesimal pyramids with their vertices at the apex of the double cone).

Now the Second Moment of the whole cylinder (of mass $6m$) about its axis is $6m \times r^2/2$. This leaves $3m^3 - 12mr^2/5 = 3mr^2/5$ as the moment of the double cone about its axis.

Hence the Second Moment about its axis of the single cone is $\frac{3}{10}mr^2$, and this will, by Art. 16, clearly hold good whether the height is equal to the radius of the base, or not. The Second Moment as to an axial plane will be $\frac{3}{20}mr^2$, and this can be extended by the principle of Art. 16 to the Second Moment of a right elliptic cone taken as to a *principal* axial plane, if r stands for the semi-axis of the base which is perpendicular to the plane of reference.

Again, since a right circular cone is a species of pyramid, we see that its Second Moment as to a plane through its apex parallel to its base is $3mh^2/5$, and as to the plane of its base $mh^2/10$, so that the axial Second Moment as to a diameter of the base is

$$(3mr^2 + 2mh^2)/20.$$

19. The idea of using Huygens' Theorem of Parallel Axes or its analogue for planar moments to find Second Moments without integration is not new—in fact, its application to the case of the rod has long been common property, and I find that there is a paper by Rehfeld in *Grunert's Archiv*, 2nd Series, Vol. 16, in which the same idea is developed and applied to most of the cases treated in the present article. So far as I am aware, however, no account of the method has hitherto been published in English. Considering this, and the fact that my treatment of the topic differs a good deal from that of Rehfeld's paper, I have ventured to offer the present communication to the Society.

[*The following paper was read 8th March 1907.*]

On Vandermonde's Theorem, and some more general Expansions.

By JOHN DOUGALL, M.A.

1. If we write

$$\binom{a}{s} \equiv \frac{a(a-1)\ldots(a-s+1)}{s\,!},$$

where a is arbitrary, but s is a positive integer, then Vandermonde's Theorem is

$$\binom{p+q}{s} = \binom{p}{s} + \binom{p}{s-1}\binom{q}{1} + \binom{p}{s-2}\binom{q}{2} + \ldots\ldots + \binom{q}{s}.$$

Divide by $\binom{p}{s}$, and put a for q, $s+\gamma$ for p.

The theorem takes the form

$$\frac{(a+\gamma+1)(a+\gamma+2)\ldots(a+\gamma+s)}{(\gamma+1)(\gamma+2)\ldots(\gamma+s)} = 1 + \frac{as}{1.\gamma+1} + \frac{a.a-1.s.s-1}{1.2.\gamma+1.\gamma+2} + \ldots (1).$$

The formula may be regarded either as summing the series on the right, or as expanding, in a particular form, the function on the left. It is by proceeding from the latter point of view that the following developments suggest themselves.

As is well known, Vandermonde's Theorem is susceptible of a great variety of proofs. For instance, in either of the forms above, it is one of the simplest examples of Newton's Interpolation formula. The proof to be given here combines in a somewhat peculiar way the principles of *symmetry* and *algebraic degree* with the *step by step* method, or method of mathematical induction.

The idea of this method obviously admitting of extension, other functions are invented, capable of expansion in finite series of factorials of suitable form. By increasing the number of terms indefinitely, various interesting infinite series are summed in terms of Π functions, in particular the hypergeometric series with fourth element unity, and several series involving the third and fourth powers of the coefficients in the expansion of $(1-x)^{-c}$.

Although hardly within the scope of the paper, one or two examples are added of a method of extending some of these summation theorems with the help of Cauchy's Theory of Residues.

2. The proof we propose to give of the theorem (1) depends mainly on this, that when α, like s, is a positive integer, the fraction on the left is symmetrical in α and s. In fact, when we multiply both numerator and denominator by $(\gamma+1)(\gamma+2)\ldots(\gamma+\alpha)$, the fraction becomes

$$\frac{(\gamma+1)(\gamma+2)\ldots(\gamma+\alpha+s)}{(\gamma+1)\ldots(\gamma+\alpha)\times(\gamma+1)\ldots(\gamma+s)},$$

the symmetry of which is obvious.

When α is zero, the fraction is unity. Now we shall define the value of a factorial $(x+1)(x+2)\ldots(x+s)$ to be unity when s is zero, and the symmetry spoken of therefore extends even to zero values of α and s.

The theorem is then true when $s=0$. Assume it true (for every α) when $s=0, 1, 2, \ldots (n-1)$. We shall prove it true for $s=n$.

For, when $s=n$, the theorem is true for $\alpha=0, 1, 2, \ldots (n-1)$, everything being symmetrical in s, α, and the theorem being true by hypothesis for $\alpha=n$ and $s=0, 1, 2, \ldots (n-1)$.

Also when $s=n$, the coefficient of the highest power of α is the same on both sides. We have therefore two rational integral functions of α of degree n, equal for n values of α and with equal coefficients of α^n. The functions are therefore identically equal; which proves the theorem.

3. The factorial function

$$\frac{(\alpha+\gamma+1)\ldots(\alpha+\gamma+s)}{(\gamma+1)\ldots(\gamma+s)}\cdot\frac{(\beta+\gamma+1)\ldots(\beta+\gamma+s)}{(\alpha+\beta+\gamma+1)\ldots(\alpha+\beta+\gamma+s)}$$

is the quotient of the function on the left of (1) by a function of the same form with $\beta+\gamma$ for γ. The same symmetry in s, α is therefore present. We shall show that the function may be expanded in the form

$$1+A_1\frac{\alpha s}{\alpha+\beta+\gamma+s}+A_2\frac{\alpha\,.\,\alpha-1\,.\,s\,.\,s-1}{\alpha+\beta+\gamma+s\,.\,\alpha+\beta+\gamma+s-1}+\ldots,$$

with the coefficients A_1, A_2, …, A_n, … independent of s and α. In the first place, if this be provisionally assumed, these coefficients can be found at once by putting $s=n$, multiplying by

$(\alpha+\beta+\gamma+1)\ldots(\alpha+\beta+\gamma+n)$, and then putting $\alpha=-\beta-\gamma-1$. Every term on the right disappears but the last, and we get

$$\frac{\beta(\beta-1)\ldots(\beta-n+1)}{(\gamma+1)\ldots(\gamma+n)}=\mathrm{A}_n\,.\,n\,!\,.$$

The theorem to be proved is then

$$\frac{(\alpha+\gamma+1)\ldots(\alpha+\gamma+s)}{(\gamma+1)\ldots(\gamma+s)}\cdot\frac{(\beta+\gamma+1)\ldots(\beta+\gamma+s)}{(\alpha+\beta+\gamma+1)\ldots(\alpha+\beta+\gamma+s)}$$

$$=1+\frac{\alpha.\beta}{1.\gamma+1}\cdot\frac{s}{\alpha+\beta+\gamma+s}+\frac{\alpha.\alpha-1.\beta.\beta-1}{1.2.\gamma+1.\gamma+2}\cdot\frac{s.s-1}{\alpha+\beta+\gamma+s.\alpha+\beta+\gamma+s-1}+\ldots \quad (2)$$

For proof, assume it true when $s=0,\ 1,\ 2,\ \ldots\ (n-1)$. Then when $s=n$, we have, after multiplication by $(\alpha+\beta+\gamma+1)..(\alpha+\beta+\gamma+n)$, two rational integral functions of α of degree n equal for the n values $0,\ 1,\ 2,\ \ldots\ (n-1)$, and the additional value $-\beta-\gamma-1$. These functions are therefore identical, and the theorem, being true for $s=0$, is true for s any positive integer.

4. The theorem (2) is noteworthy for two reasons. The chief is that it contains as a limiting case the highly important summation of the hypergeometric series with fourth element unity. To obtain this, take the limit of the two members of (2) for s infinite.

The left hand member may be written

$$\frac{(\alpha+\gamma+1)\ldots(\alpha+\gamma+s)}{s\,!}s^{-\alpha-\gamma}\cdot\frac{s\,!}{(\gamma+1)\ldots(\gamma+s)}s^{\gamma}\,.$$

$$\frac{(\beta+\gamma+1)\ldots(\beta+\gamma+s)}{s\,!}s^{-\beta-\gamma}\cdot\frac{s\,!}{(\alpha+\beta+\gamma+1)\ldots(\alpha+\beta+\gamma+s)}s^{\alpha+\beta+\gamma}.$$

But according to Gauss's definition of the Π function,

$$\Pi z=\underset{s=\infty}{\mathrm{Limit}}\,\frac{s\,!}{(z+1)(z+2)\ldots(z+s)}s^{z},$$

so that the limit of the left hand member of (2) is

$$\frac{\Pi\gamma\,.\,\Pi(\alpha+\beta+\gamma)}{\Pi(\alpha+\gamma)\,\Pi(\beta+\gamma)}.$$

(It may be noted here, in connection with the Π function that the definition gives at once

$$(z+1)(z+2)\ldots(z+n)=\frac{\Pi(n+z)}{\Pi z},$$

so that the definition itself may be read

$$\operatorname*{Lt}_{s=\infty} \frac{\Pi s}{\Pi(s+z)} s^z = 1,$$

$$\text{or} \quad (z+1)(z+2)\ldots(z+s) = \frac{1}{\Pi z} s^z \Pi s, \quad \text{asymptotically.)}$$

Hence the *limit* of the series on the right of (2) has this value, whatever be the relative values of α, β, γ; this limit we can easily show to be equal to the series of limits of the individual terms in order, provided the real part of $\alpha+\beta+\gamma$ is not negative.

For under this restriction the moduli of the factors

$$\frac{s}{\alpha+\beta+\gamma+s}, \; \frac{s-1}{\alpha+\beta+\gamma+s-1}, \ldots .$$

cannot exceed unity, whatever be the value of s. Hence the residue after n terms of the series of moduli of the terms of (2) cannot exceed the corresponding residue in the absolutely converging series

$$1 + \frac{\alpha . \beta}{1 . \gamma+1} + \frac{\alpha . \alpha-1 . \beta . \beta-1}{1 . 2 . \gamma+1 . \gamma+2} + \ldots .$$

But we can choose n, independent of s, so that the latter residue is as small as we please, and, having so chosen n, we can then make s infinite.

Hence

$$1 + \frac{\alpha . \beta}{1 . \gamma+1} + \frac{\alpha . \alpha-1 . \beta . \beta-1}{1 . 2 . \gamma+1 . \gamma+2} + \ldots = \frac{\Pi\gamma . \Pi(\alpha+\beta+\gamma)}{\Pi(\alpha+\gamma)\,\Pi(\beta+\gamma)} . \qquad (3)$$

The equation (3) which we have proved subject to the restriction $R(\alpha+\beta+\gamma) \nless 0$, holds, in fact, so long as the series converges, that is, so long as $R(\alpha+\beta+\gamma) > -1$.

The equation is obviously equivalent to the more familiar form

$$F(\alpha, \beta, \gamma, 1) \equiv$$

$$1 + \frac{\alpha . \beta}{1 . \gamma} + \frac{\alpha . \alpha+1 . \beta . \beta+1}{1 . 2 . \gamma . \gamma+1} + \ldots = \frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} .$$

Vandermonde's Theorem is the special case of (3) for one of α, β a positive integer.

5. A second point of interest about the equation (2) is that with its aid a proof by actual multiplication can be given of the theorem

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x). \qquad (4)$$

For if we multiply

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha . \alpha + 1 . \beta . \beta + 1}{1 . 2 . \gamma . \gamma + 1} x^2 + . .$$

by

$$(1 - x)^{-(\gamma - \alpha - \beta)} = 1 + \frac{\gamma - \alpha - \beta}{1} x + \frac{\gamma - \alpha - \beta . \gamma - \alpha - \beta + 1}{1 . 2} x^2 + \dots$$

and arrange the product by powers of x, as we are entitled to do, if $|x| < 1$, the coefficient of x^n is

$$\frac{(\gamma - \alpha - \beta)..(\gamma - \alpha - \beta + n - 1)}{n!}$$

$$\left\{1 + \frac{\alpha . \beta}{1 . \gamma} \frac{n}{\gamma - \alpha - \beta + n - 1} + \frac{\alpha . \alpha + 1 . \beta . \beta + 1 . n . n - 1}{1.2.\gamma.\gamma + 1 . \gamma - \alpha - \beta + n - 1 . \gamma - \alpha - \beta + n - 2} + \dots\right\}$$

$$= \frac{(\gamma - \alpha - \beta)..(\gamma - \alpha - \beta + n - 1)}{n!} \cdot \frac{(\gamma - \alpha)..(\gamma - \alpha + n - 1)}{\gamma \dots (\gamma + n - 1)} \cdot \frac{(\gamma - \beta) \dots (\gamma - \beta + n - 1)}{(\gamma - \alpha - \beta)..(\gamma - \alpha - \beta + n - 1)},$$

as we find from (2) by changing α into $-\alpha$, β into $-\beta$, γ into $\gamma - 1$, and s into n.

This is the same as the coefficient of x^n in $F(\gamma - \alpha, \gamma - \beta, \gamma, x)$, whence the theorem.

It is also worthy of remark that just as (2) has been used to prove (4), so Vandermonde's Theorem (1) may be used to prove the equally important relation

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma, \frac{x}{x - 1}\right). \qquad (5)$$

Having found (5) in this way, we might deduce (4) in either of two ways; (i), by applying to $F\left(\gamma - \beta, \alpha, \gamma, \frac{x}{x - 1}\right)$ the same transformation as is applied to $F(\alpha, \beta, \gamma, x)$ in (5), thus finding

$$F\left(\alpha, \gamma - \beta, \gamma, \frac{x}{x - 1}\right) = \left(1 - \frac{x}{x - 1}\right)^{\beta - \gamma} F(\gamma - \beta, \gamma - \alpha, \gamma, x);$$

or (ii), simply from the symmetry of $F(\alpha, \beta, \gamma, x)$ in α, β by deriving from (5)

$$(1 - x)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma, \frac{x}{x - 1}\right) = (1 - x)^{-\beta} F\left(\beta, \gamma - \alpha, \gamma, \frac{x}{x - 1}\right),$$

which is merely another way of writing (4).

By using the latter method, it will be seen that through the medium of these hypergeometric series we might derive (2) from (1).

6. In proving (1) and (2) we have taken advantage of two properties of such a factorial function as $\dfrac{(x+a+1)\dots(x+a+s)}{(a+1)\dots(a+)s}$, namely, that it is of degree s in x, and that when x, like s, is a positive integer, it is symmetrical in s, x. By joining to these a third property of factorials we can invent a much more general expansion.

The new property is that the product of two factorials

$$(x+a+1)\dots(x+a+s) \text{ and } (x+b+1)\dots(x+b+s)$$

is not altered by the substitution of $-x-a-b-s-1$ for x.
In fact obviously $(x+p)(x+q)$ remains unchanged when $-x-p-q$ is written for x, and the factors of the above product can be arranged in pairs

$$(x+a+1)(x+b+s),\ (x+a+2)(x+a+b+s-1), \dots$$

of the form $(x+p)(x+q)$, with $p+q$ the same for all, namely, equal to $a+b+s+1$.

Consider now the function

$$\frac{(x+a+1)\dots(x+a+s)\,.\,(x+b+1)\dots(x+b+s)}{(x+c+1)\dots(x+c+s)\,.\,(x+d+1)\dots(x+d+s)}\cdot\frac{(c+1)\dots(c+s)\,.\,(d+1)\dots(d+s)}{(a+1)\dots(a+s)\,.\,(b+1)\dots(b+s)}.$$

This has clearly the property of symmetry in x, s when x is a positive integer. It will also have the property of remaining unchanged when $-x-a-b-s-1$ is put for x, provided $a+b=c+d$.

It can be expanded in a series of functions of x, s, all of which possess these two properties, namely, in the form

$$1+\mathrm{A}_1\frac{x\,.\,s\,.\,x+a+b+s+1}{x+c+1\,.\,s+c+1\,.\,x+d+s}$$

$$+\mathrm{A}_2\frac{x\,.\,x-1\,.\,s\,.\,s-1\,.\,x+a+b+s+1\,.\,x+a+b+s+2}{x+c+1\,.\,x+c+2\,.\,s+c+1\,.\,s+c+2\,.\,x+d+s\,.\,x+d+s-1}+\dots;$$

where the A's are independent of s and x.

The proof is practically the same as in § 3. Thus, assuming the theorem for a moment, we determine A_n by giving s the value n, multiplying by

$$(x+c+1)\dots(x+c+n)\,.\,(x+d+n)\dots(x+d+1),$$

and putting $-d-1$ (or $-c-n$) for x.

We thus find

$$\mathrm{A}_n \,.\, (-1)^n n! \frac{c+n \,.\, c+n+1 \ldots c+2n-1}{c+n+1 \ldots\ldots\ldots\ldots\ldots c+2n}$$

$$= (a-d)\ldots(a-d+n-1)\,.\,(b-d)\ldots(b-d+n-1)\,.\,\frac{(c+1)\ldots(c+n)}{(a+1)\ldots(a+n).(b+1)\ldots(b+n)}$$

With A_n thus determined, assume the theorem (which is true for $s=0$) for $s=0, 1, 2, \ldots (n-1)$. It will be true for $s=n$. For when $s=n$, it is, by hypothesis and the symmetry, true for $x=0, 1, 2, \ldots (n-1)$; also for $x = -a-b-n-1,\ -a-b-n-2, \ldots$ other n values, since no term is changed by changing x into $-x-a-b-n-1$; further when we multiply by $(x+c+1)\ldots(x+c+n)\,.\,(x+d+1)\ldots(x+d+n)$, the theorem is true for $x = -d-1$, $2n+1$ values in all.

The theorem asserts the equality of two rational integral functions of x of degree $2n$. These being equal for $2n+1$ values of x are equal identically, whence the theorem.

7. The symmetry of the formula just found will be more apparent if we write $y+c$ for a, $z+c$ for b, and therefore $y+z+c$ for d. We may also, for the sake of compactness, introduce the notation

$$a^{(s)} = a(a+1)\ldots(a+s-1), \qquad a^{(-s)} = a(a-1)\ldots(a-s+1).$$

We thus obtain

$$\frac{(y+z+c+1)^{(s)}(z+x+c+1)^{(s)}(x+y+c+1)^{(s)}(c+1)^{(s)}}{(x+c+1)^{(s)}(y+c+1)^{(s)}(z+c+1)^{(s)}(x+y+z+c+1)^{(s)}}$$

$$= 1 - \frac{(c+2)}{c}\,.\,\frac{c}{1}\,.\,\frac{s}{s+c+1}\,.\,\frac{x}{x+c+1}\,.\,\frac{y}{y+c+1}\,.\,\frac{z}{z+c+1}\,.\,\frac{x+y+z+2c+s+1}{x+y+z+c+s} + \ldots$$

$$= \sum_{n=0}^{n=s} (-1)^n \frac{c+2n}{c} \frac{c^{(n)}}{n!} \frac{s^{(-n)}}{(s+c+1)^{(n)}} \frac{x^{(-n)}}{(x+c+1)^{(n)}} \frac{y^{(-n)}}{(y+c+1)^{(n)}} \frac{z^{(-n)}}{(z+c+1)^{(n)}} \frac{(x+y+z+2c+s+1)^{(n)}}{(x+y+z+c+s)^{(-n)}}. \qquad (6)$$

If we take c as a function of x, y, z, s and a new variable t, defined by the equation

$$x+y+z+2c+s+1=-t$$

so that
$$c=-\tfrac{1}{2}(x+y+z+s+t+1), \qquad \cdot \quad \cdot \quad (7)$$

then the series on the right of (6) has its general term symmetrical in x, y, z, s, t. It is easy to see that the function on the left is also symmetrical in x, y, z, t, for interchange of x and t is equivalent to the substitution of $-x-y-z-2c-s-1$ for x.

It may be pointed out that there is another set of substitutions under which the function on the left of (6) is invariant; *e.g.*, $-y$, $-z$, $y+z+c$ for y, z, c respectively. If the value (7) of c is first substituted, this is equivalent to change of sign of y, z only. The signs of x, t may also afterwards be changed; the result is the same as that found by writing $-x$, $-y$, $-z$, $-c-s-1$ for x, y, z, c in the series (6). We have thus altogether three different types of expansion of the function on the left of (6).

If we take $x+y+z+c=0$, s disappears from the general term of the series (6), and the function on the left gives the sum of $s+1$ terms of the series, as may easily be verified by the method of differences.

8. The formula (6), involving five variables, contains, of course, a large number of special or limiting cases. For example, by writing $u-z$ for c, and making z infinite, we deduce (2), which itself, when β is made infinite, gives (1).

But the most interesting results are those obtained by making the positive integer s infinite, either with or without previous change of variables.

First take (6) as it stands.

Substituting their asymptotic values for the factorials (§ 4), we find for the limit of the function on the left

$$\frac{\Pi(x+c)\,\Pi(y+c)\,\Pi(z+c)\,\Pi(x+y+z+c)}{\Pi(y+z+c)\,\Pi(z+x+c)\,\Pi(x+y+c)\,\Pi c}\,.$$

As for the series, we shall show that its limit can be found term by term, as in the case of the series (2), subject to a certain restriction on the values of the variables.

The series of term by term limits is

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{c+2n}{c}\frac{c^{(n)}}{n!}\frac{x^{(-n)}}{(x+c+1)^{(n)}}\frac{y^{(-n)}}{(y+c+1)^{(n)}}\frac{z^{(-n)}}{(z+c+1)^{(n)}},$$

in which the general term is of order $n^{-(2x+2y+2z+2c+3)}$, so that the series converges absolutely if $R(x+y+z+c)>-1$.

But if we write σ for $x+y+z+c$, the factor containing s in the general term of (6) is

$$\frac{s\ \ldots\ldots(s-n+1)}{(s+c+1)\ldots(s+c+n)}\cdot\frac{(s+\sigma+c+1)\ldots(s+\sigma+c+n)}{(s+\sigma)\quad\ldots(s+\sigma-n+1)}. \tag{8}$$

This is got from the corresponding factor of the previous term by multiplying by

$$\frac{(s-n+1)(s+\sigma+c+n)}{(s+c+n)(s+\sigma-n+1)}\quad\text{or}\quad\frac{1+\dfrac{\sigma}{s+c+n}}{1+\dfrac{\sigma}{s-n+1}},$$

which, supposing for brevity that c and σ are real, is not greater than 1 if σ is not negative and

$$s+c+n>s-n+1,\ \text{that is,}\ n>\tfrac{1}{2}(1-c).$$

Under these conditions, if N be any positive integer greater than $\frac{1}{2}(1-c)$, the value of the factor (8) for any n greater than N cannot exceed its value for $n=\mathrm{N}$, which has obviously a finite upper limit independent of s.

The residue after n terms in (6) cannot therefore exceed a certain finite multiple of the corresponding residue in the series of limits. Hence, as in § 4, we can take the limit of the series in (6) term by term, and obtain

$$\frac{\Pi(x+c)\,\Pi(y+c)\,\Pi(z+c)\,\Pi(x+y+z+c)}{\Pi(y+z+c)\,\Pi(z+x+c)\,\Pi(x+y+c)\,\Pi c}$$

$$=\sum_{n=0}^{n=\infty}(-1)^n\frac{c+2n}{c}\frac{c^{(n)}}{n!}\frac{x^{(-n)}}{(x+c+1)^{(n)}}\frac{y^{(-n)}}{(y+c+1)^{(n)}}\frac{z^{(-n)}}{(z+c+1)^{(n)}}. \tag{9}$$

The theorem has been proved for $x+y+z$ and c real, and $x+y+z+c>-1$. It is, in fact, true if only $R(x+y+z+c)>-1$.

For the extension we may be content to rely on the general Theorem of Continuation in the Theory of Functions.

9. Suppose we consider the function expanded in (9) as a function of the complex variable x. The infinities of the function are those of $\Pi(x+c)$ and $\Pi(x+y+z+c)$. The series represents the function over part of the x plane, and in this part the first set of infinities, those of $\Pi(x+c)$, are indicated by individual terms becoming infinite. It is worth remarking, as the circumstances often occur, that by a simple change of variable, we may obtain an expansion indicating the other set of infinities. We have only to write $-y$, $-z$, $c+y+z$ for y, z, c respectively.

For another instance of the point, in the ordinary hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}+\frac{\alpha.\alpha+1.\beta.\beta+1}{1.2.\gamma.\gamma+1}+\ldots=\frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)},$$

change α, β, γ into $-\alpha$, $-\beta$, $\gamma-\alpha-\beta$ respectively.

10. Some special cases of (9) may be set down.

In (9) put $c=0$. Then

$$\frac{\Pi x\,\Pi y\,\Pi z\,\Pi(x+y+z)}{\Pi(y+z)\,\Pi(z+x)\,\Pi(x+y)}$$

$$=1-2\frac{x}{x+1}\cdot\frac{y}{y+1}\cdot\frac{z}{z+1}+2\frac{x.x-1}{x+1.x+2}\cdot\frac{y.y-1}{y+1.y+2}\cdot\frac{z.z-1}{z+1.z+2}-\ldots,$$

$$R(x+y+z)>-1. \qquad (10)$$

In (10) let z go to infinity through real positive values. Thus

$$\frac{\Pi x\,\Pi y}{\Pi(x+y)}=1-2\frac{x}{x+1}\cdot\frac{y}{y+1}+2\frac{x.x-1}{x+1.x+2}\cdot\frac{y.y-1}{y+1.y+2}-\ldots,$$

$$R(x+y)\not<-\tfrac{1}{2}. \qquad (11)$$

In (10) take $z=-\frac{1}{2}$, and get

$$\frac{\Pi x\,\Pi y\,\Pi(-\frac{1}{2})\,\Pi(x+y-\frac{1}{2})}{\Pi(x-\frac{1}{2})\,\Pi(y-\frac{1}{2})\,\Pi(x+y)}$$

$$=1+2\frac{x}{x+1}\cdot\frac{y}{y+1}+2\frac{x.x-1}{x+1.x+2}\cdot\frac{y.y-1}{y+1.y+2}+\ldots,$$

$$R(x+y)>-\tfrac{1}{2}. \qquad (\Pi(-\tfrac{1}{2})=\sqrt{\pi}). \qquad (12)$$

In (12) put $y=+\infty$. Thus

$$\frac{\Pi x\,\Pi(-\frac{1}{2})}{\Pi(x-\frac{1}{2})}=1+2\frac{x}{x+1}+2\frac{x.x-1}{x+1.x+2}+\ldots, \qquad (13)$$

$$R(x)\not<0.$$

Interesting results are obtained by making each of the variables half an odd integer in the last four formulæ. For instance, $x=\frac{1}{2}$ in (13) gives Gregory's series for π.

11. In (9) put $z=-\frac{1}{2}c$. Thus

$$\sum_{n=0}^{n=\infty} \frac{c^{(n)}}{n!}\frac{x^{(-n)}}{(x+c+1)^{(n)}}\frac{y^{(-n)}}{(y+c+1)^{(n)}}$$

$$=\frac{\Pi(x+c)\,\Pi(y+c)\,\Pi\frac{c}{2}\,\Pi\left(x+y+\frac{c}{2}\right)}{\Pi\left(x+\frac{c}{2}\right)\Pi\left(y+\frac{c}{2}\right)\Pi(x+y+c)\,\Pi c}, \quad - \quad - \quad (14)$$

$$R\left(x+y+\frac{c}{2}\right)>-1.$$

In (14) put $x=y=-c$. Hence

$$\sum_{n=0}^{n=\infty}\left\{\frac{c(c+1)\dots(c+n-1)}{n!}\right\}^3=\frac{\Pi\frac{c}{2}\,\Pi\left(-\frac{3c}{2}\right)}{\Pi\left(\frac{c}{2}\right)\Pi\left(-\frac{c}{2}\right)\Pi(-c)\,\Pi c}$$

$$=\cos\frac{\pi c}{2}\,\frac{\Pi\left(-\frac{3c}{2}\right)}{\left\{\Pi\left(-\frac{c}{2}\right)\right\}^3}, \quad - \quad (15)$$

$$\left(\text{since } \Pi a\,\Pi(-a)=\frac{\pi a}{\sin\pi a}\right); \quad R(c)<\tfrac{2}{3}.$$

This gives the sum of the cubes of the coefficients of the expansion of $(1-x)^{-c}$. For a proof by means of the theory of hypergeometric functions see Dixon, "Summation of certain Series," *Proc. Lond. Math. Soc.*, Vol. 35, page 284.

In (9) make z infinite, x and $y=-c$.

Then

$$\sum_{n=0}^{n=\infty}(-1)^n\frac{c+2n}{c}\left\{\frac{c(c+1)\dots(c+n-1)}{n!}\right\}^3=\frac{\sin\pi c}{\pi c},$$

$$c \not> \tfrac{1}{3}. \quad - \quad - \quad - \quad - \quad - \quad (16)$$

In (9) make $x=y=z=-c$. Then

$$\sum_{n=0}^{n=\infty}\frac{c+2n}{c}\left\{\frac{c(c+1)\dots(c+n-1)}{n!}\right\}^4=\frac{\sin\pi c}{\pi c}\,\frac{\Pi(-2c)}{\{\Pi(-c)\}^2},$$

$$c<\tfrac{1}{2}. \quad - \quad - \quad - \quad - \quad - \quad (17)$$

In (9) make $x=y=-c$ and $z+(z+c+1)=0$ or $z=-\dfrac{c+1}{2}$.

Then

$$\frac{\sin\pi c}{\pi c}\,\frac{\Pi\left(\frac{c-1}{2}\right)\Pi\left(-\frac{3c+1}{2}\right)}{\left\{\Pi\left(-\frac{c+1}{2}\right)\right\}^2}$$

$$=\sum_{n=0}^{n=\infty}\frac{c+2n}{c}\left(\frac{c\,.\,c+1\,\ldots\,c+n-1}{n\,!}\right)^3 \quad . \quad (18)$$

12. Another limiting case of the fundamental theorem (6) deserves attention, less perhaps for the result itself than for the unusual way in which one has to go to the limit to obtain it. The example illustrates the necessity of caution before taking the limit of a series term by term, even when a definite limit exists, and the series of term by term limits converges.

In (6) put $c=a-s$ and let s increase indefinitely.

Consider first the sum of the series.

We have

$$\begin{aligned}(c+1)\ldots(c+s) &= (a-s+1)\ldots a\\ &= (-1)^s(s-a-1)\ldots(-a)\\ &= (-1)^s\Pi(s-a-1)/\Pi(-a-1)\\ &= (-1)^s s^{-a-1}\Pi s/\Pi(-a-1),\ \text{asymptotically,}\end{aligned}$$

and similarly with the other factorials on the left of (6), the limit of which is therefore

$$\frac{\Pi(-x-a-1)\,\Pi(-y-a-1)\,\Pi(-z-a-1)\,\Pi(-x-y-z-a-1)}{\Pi(-y-z-a-1)\,\Pi(-z-x-a-1)\,\Pi(-x-y-a-1)\,\Pi(-a-1)}. \quad (18^a)$$

Next take the series itself. This is

$$\sum_{n=0}^{n=s}(-1)^n\frac{a-s+2n}{a-s}\,\frac{(a-s)^{(n)}}{n\,!}\,\frac{s^{(-n)}}{(a+1)^{(n)}}\,\frac{x^{(-n)}}{(x+a-s+1)^{(n)}}\ \text{do. } y,\ z,\,.$$

$$\frac{(x+y+z+2a-s+1)^{(n)}}{(x+y+z+a)^{(-n)}} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(the contraction do. y, z, signifying that two factors have to be put in, the same functions of y, z respectively as the immediately preceding factor is of x), and the series of term by term limits is

$$\sum_{n=0}^{n=\infty}\frac{x^{(-n)}\,y^{(-n)}\,z^{(-n)}}{n\,!\,(a+1)^{(n)}\,(x+y+z+a)^{(-n)}}, \quad . \quad . \quad (20)$$

a series of the same form as that in (2), converging absolutely except for such values of the variables as make the terms themselves infinite. But we can see at a glance that (18^a) and (20) are not equivalent expressions; taken as functions of a, for example, their infinities are quite different. The explanation is, in broad terms, that as s increases it is the central part of the series (19) and not the part towards the end, that dwindles into insignificance and has, so to speak, to be sent to infinity when we go to the limit.

Thus the last term of (19) is

$$(-1)^s \frac{a+s}{a-s} \frac{(a-s)\ldots(a-1)}{(a+1)\ldots(a+s)} \frac{x\ldots(x-s+1)}{(x+a-s+1)\ldots(x+a)} \text{ do. } y, z, . \frac{(x+y+z+2a-s+1)\ldots(x+y+z+2a)}{(x+y+z+a)\ldots(x+y+z+a-s+1)}$$

$$= \lambda \text{ say,}$$

which has the finite limit

$$-\frac{\Pi a}{\Pi(-a)} \frac{\Pi(-x-a-1)}{\Pi(-x-1)} \text{ do. } y, z, . \frac{\Pi(-x-y-z-a-1)}{\Pi(-x-y-z-2a-1)}. \qquad (21)$$

Now write the series (19) with $p+1$ terms in direct order, and the remaining $s-p$ terms in reverse order. As the equivalent of (19) we thus get

$$\sum_{n=0}^{n=p} \frac{s-2n-a}{s-a} \frac{s^{(-n)}}{n!} \frac{(s-a)^{(-n)}}{(a+1)^{(n)}} \frac{x^{(-n)}}{(s-x-a-1)^{(-n)}} \text{ do. } y, z, . \frac{(s-x-y-z-2a-1)^{(-n)}}{(x+y+z+a)^{(-n)}}$$

$$-\lambda . \sum_{n=0}^{n=s-p-1} \frac{s-2n+a}{s+a} \frac{s^{(-n)}}{n!} \frac{(s+a)^{(-n)}}{(-a+1)^{(n)}} \frac{(x+a)^{(-n)}}{(s-x-1)^{(n)}} \text{ do. } y, z, . \frac{(s-x-y-z-a-1)^{(-n)}}{(x+y+z+2a)^{(-n)}}, \qquad (22)$$

In the term under the first Σ, the factor containing s is

$$\frac{s-2n-a}{s-a} \frac{s^{(-n)}(s-a)^{(-n)}(s-x-y-z-2a-1)^{(-n)}}{(s-x-a-1)^{(-n)} \text{ do. } y, z.}. \qquad (23)$$

We can define the integer p as a function of s to suit our convenience. All that is necessary for the following argument is that, as s increases, the ratio $s/(s-p)$ should be contained between finite limits. But we may make

the simplest hypothesis, namely, that $p = s - p - 1$, which requires that s should increase through odd integral values, and that $p = \dfrac{s-1}{2}$.

Then in (22) the greatest value of n is $\dfrac{s-1}{2}$, so that if s is large, $s - n$ is also large. Hence by the asymptotic expression for a Π function, we find that the value of (23) is approximately $\dfrac{s-2n-a}{s-a}\left(\dfrac{s}{s-n}\right)^2$, and therefore clearly lies between finite limits independent of s, as n goes from 0 to $\frac{1}{2}(s-1)$.

The term under the second Σ in (22) is of the same form as that under the first, with variables changed.

Hence, by the argument used immediately before equation (9), the limit of (22) can be taken term by term.

We have therefore

$$\sum_{n=0}^{n=\infty} \frac{x^{(-n)} y^{(-n)} z^{(-n)}}{n\,!\,(a+1)^{(n)}(x+y+z+a)^{(-n)}} = \frac{\Pi a\, \Pi(-x-a-1)\, \Pi(-y-a-1)\, \Pi(-z-a-1)\, \Pi(-x-y-z-a-1)}{\Pi(-a)\, \Pi(-x-1)\, \Pi(-y-1)\, \Pi(-z-1)\, \Pi(-x-y-z-2a-1)}.$$

$$\sum_{n=0}^{n=\infty} \frac{(x+a)^{(-n)}(y+a)^{(-n)}(z+a)^{(-n)}}{n\,!\,(-a+1)^{(n)}(x+y+z+2a)^{(-n)}} = \frac{\Pi(-x-a-1)\Pi(-y-a-1)\Pi(-z-a-1)\Pi(-x-y-z-a-1)}{\Pi(-y-z-a-1)\Pi(-z-x-a-1)\Pi(-x-y-a-1)\Pi(-a-1)}. \qquad (24)$$

This reduces to (2) when z is a positive integer.

Write $-x-c$, $-y-c$, $-z-c$, $2c$ for x, y, z, a respectively, and (24) takes the neater form

$$\frac{1}{\Pi(2c)\,\Pi(x-c-1)\,\Pi(y-c-1)\Pi(z-c-1)\Pi(x+y+z+c-1)} \sum_{n=0}^{n=\infty} \frac{(x+c)^{(n)}(y+c)^{(n)}(z+c)^{(n)}}{n\,!\,(2c+1)^{(n)}(x+y+z+c)^{(n)}}$$

$$- \frac{1}{\Pi(-2c)\,\Pi(x+c+1)\,\Pi(y+c+1)\Pi(z+c+1)\Pi(x+y+z-c-1)} \sum_{n=0}^{n=\infty} \frac{(x-c)^{(n)}(y-c)^{(n)}(z-c)^{(n)}}{n\,!\,(-2c+1)^{(n)}(x+y+z-c)^{(n)}}$$

$$= -\frac{1}{\Pi(y+z-1)\,\Pi(z+x-1)\Pi(x+y-1)} \frac{\sin 2\pi c}{\pi}. \qquad (25)$$

13. For the sake of pointing out some natural extensions of the preceding results, we will now abandon the purely elementary methods hitherto used, and take advantage of the powerful weapon furnished by the Theory of Complex Integration and Residues.

As one of the simplest subjects for the extension contemplated, take the ordinary hypergeometric series

$$1+\frac{\alpha\beta}{1.\gamma}+\frac{\alpha.\alpha+1.\beta.\beta+1}{1.2.\gamma.\gamma+1}+\dots.$$

This may be written

$$\frac{\Pi(\gamma-1)}{\Pi(\alpha-1)\Pi(\beta-1)}\sum_{n=0}^{n=\infty}\frac{\Pi(\alpha-1+n)\Pi(\beta-1+n)}{\Pi n\,\Pi(\gamma-1+n)}.$$

The sum of the series is known; we proceed to find the sum of the more general series

$$S=\sum_{n=-\infty}^{n=\infty}\frac{\Pi(a+n)\,\Pi(b+n)}{\Pi(c+n)\,\Pi(d+n)}, \qquad - \quad - \quad (26)$$

which reduces to the preceding when c or d is zero, or an integer.

Consider the function of the complex variable z,

$$f(z)=\frac{\Pi(a+z)\,\Pi(b+z)}{\Pi(c+z)\,\Pi(d+z)}\,\frac{\pi\cos\pi z}{\sin\pi z}. \qquad - \quad - \quad - \quad (27)$$

This function has three sets of simple infinities, namely, those of $\Pi(a+z)$, $\Pi(b+z)$ and $1/\sin\pi z$; the series S is obviously the sum of the residues of the function at the poles of $1/\sin\pi z$.

It may be proved that, subject to a restriction on the values of a, b, c, d, the sum of the residues of $f(z)$ at all its poles is zero.

To prove this, we have to consider the form of the uniform function $f(z)$ for z infinite.

Now, when z tends to infinity, we have the asymptotic expression

$$\Pi(z)=\sqrt{2\pi}\,e^{(z+\frac{1}{2})\log z-z}, \qquad - \quad - \quad - \quad (28)$$

valid on the supposition that the argument or phase of z lies between $-\pi$ and π. If the real part of z is large and negative, but its imaginary part is not large, the form (28) fails; but in this case the asymptotic form of $\Pi(z)$ is easily deduced from the fundamental relation

$$\Pi z\,\Pi(-z)=\pi z/\sin\pi z. \qquad - \quad - \quad - \quad (29)$$

From (28) it follows easily that, asymptotically,

$$\Pi(z+a)=z^{a}\,\Pi(z).$$

Hence, unless the phase of z tends to $\pm\pi$, we have

$$\frac{\Pi(a+z)\,\Pi(b+z)}{\Pi(c+z)\,\Pi(d+z)} = z^{a+b-c-d}, \text{ asymptotically.}$$

In the excepted case, write $z = -\zeta$; then

$$\Pi(a+z) = \Pi(a-\zeta)$$
$$= \frac{1}{\Pi(\zeta-a)}\,\frac{\pi(\zeta-a)}{\sin\pi(\zeta-a)},$$

and $\dfrac{\Pi(a+z)\,\Pi(b+z)}{\Pi(c+z)\,\Pi(d+z)} = \zeta^{a+b-c-d}\,\dfrac{\sin\pi(z+c)\sin\pi(z+d)}{\sin\pi(z+a)\sin\pi(z+b)}$, asymptotically.

It may easily be proved that we can, avoiding any zero of $\sin\pi(z+a)$ or $\sin\pi(z+b)$ or $\sin\pi z$, describe a circle in the z plane of radius r as large as we please, along which

$$\left|\frac{\sin\pi(z+c)\,\sin\pi(z+d)\,\cos\pi z}{\sin\pi(z+a)\,\sin\pi(z+b)\,\sin\pi z}\right|$$

has a finite upper limit independent of r.

Also the integral $\int |z^{a+b-c-d}dz|$ taken along the circle will tend to zero as r increases, provided the real part of $a+b-c-d$ is less than -1.

Hence if $R(a+b-c-d) < -1$, the sum of the residues of $f(z)$ is zero.

The residue of $\Pi(z+a)$ at the pole $z = -a-p-1$ $(p=0, 1, \dots)$ is $(-1)^p/\Pi p$.

The residue of $f(z)$ at this pole is therefore

$$\frac{(-1)^p}{\Pi p}\,\frac{\Pi(b-a-p-1)}{\Pi(c-a-p-1)\,\Pi(d-a-p-1)}\,(-\pi\cot\pi a),$$

and the sum of the residues at the poles of $\Pi(a+z)$ is

$$-\pi\cot\pi a\,\frac{\Pi(b-a-1)}{\Pi(c-a-1)\,\Pi(d-a-1)}$$
$$\left\{1+\frac{a-c+1\,.\,a-d+1}{1\,.\,a-b+1}+\frac{a-c+1\,.\,a-c+2\,.\,a-d+1\,.\,a-d+2}{1\,.\,2\,.\,a-b+1\,.\,a-b+2}+\dots\right\}$$
$$= -\pi\cot\pi a\,\frac{\Pi(b-a-1)}{\Pi(c-a-1)\,\Pi(d-a-1)}\cdot\frac{\Pi(a-b)\,\Pi(c+d-a-b-2)}{\Pi(c-b-1)\,\Pi(d-b-1)}$$
$$= \pi^2\,\frac{\cot\pi a}{\sin\pi(a-b)}\,\frac{\Pi(c+d-a-b-2)}{\Pi(c-a-1)\,\Pi(d-a-1)\,\Pi(c-b-1)\,\Pi(d-b-1)}.$$

For the sum of the residues at the poles of $\Pi(b+z)$ we have only to interchange a and b in this.

Hence

$$S = -\text{ sum of residues at poles of } \Pi(z+a) \text{ and } \Pi(z+b)$$

$$= \frac{\pi^2}{\sin\pi a \sin\pi b} \frac{\Pi(c+d-a-b-2)}{\Pi(c-a-1)\,\Pi(d-a-1)\,\Pi(c-b-1)\Pi(d-b-1)}. \tag{30}$$

We may write

$$S = \sum_{n=0}^{n=\infty} + \sum_{n=-1}^{n=-\infty}$$

$$= \frac{\Pi a\,\Pi b}{\Pi c\,\Pi d}\left(\begin{array}{l} 1 + \dfrac{a+1\,.\,b+1}{c+1\,.\,d+1} + \dfrac{a+1\,.\,a+2\,.\,b+1\,.\,b+2}{c+1\,.\,c+2\,.\,d+1\,.\,d+2} + \dots \\ + \quad \dfrac{cd}{ab} \quad + \quad \dfrac{c\,.\,c-1\,.\,d\,.\,d-1}{a\,.\,a-1\,.\,b\,.\,b-1} \quad + \dots \end{array}\right).$$

For a, b write $-a-1$, $-b-1$ and the result becomes

$$1 + \frac{ab}{c+1\,.\,d+1} + \frac{a\,.\,a-1\,.\,b\,.\,b-1}{c+1\,.\,c+2\,.\,d+1\,.\,d+2} + \dots$$

$$+ \frac{cd}{a+1\,.\,b+1} + \frac{c\,.\,c-1\,.\,d\,.\,d-1}{a+1\,.\,a+2\,.\,b+1\,.\,b+2} + \dots$$

$$= \frac{\Pi a\,\Pi b\,\Pi c\,\Pi d\,\Pi(a+b+c+d)}{\Pi(a+c)\,\Pi(a+d)\,\Pi(b+c)\,\Pi(b+d)}. \tag{31}$$

$$R(a+b+c+d) > -1.$$

This reduces to (3) when d is zero.

14. The result (9), which reduces to the summation of the hypergeometric series, just generalised, when we put $z = u - c$ and make c infinite, may be extended by a similar process. After putting $x-c$, $y-c$, $z-c$ for x, y, z in (9), we may write it in the form

$$\sum_{n=0}^{n=\infty} (c+2n) \frac{\Pi(c-1+n)}{\Pi n} \frac{\Pi(c-1-x+n)}{\Pi(x+n)} \text{ do. } y, z$$

$$= \frac{\Pi(c-x-1)\,\Pi(c-y-1)\,\Pi(c-z-1)\,\Pi(x+y+z-2c)}{\Pi(y+z-c)\Pi(z+x-c)\,\Pi(x+y-c)}. \tag{32}$$

With the help of this theorem and the method of residues, we can sum the more general series

$$S = \sum_{n=-\infty}^{n=\infty} (c+2n) \frac{\Pi(c-1-t+n)}{\Pi(t+n)} \text{ do. } x, y, z, \tag{33}$$

which reduces to the series of (32) when $t = 0$.

Take the uniform function of ζ,

$$f(\zeta) = (c+2\zeta)\frac{\Pi(c-1-t+\zeta)}{\Pi(t+\zeta)} \text{ do. } x, y, z, . \frac{\pi\cos\pi\zeta}{\sin\pi\zeta}.$$

The sum of the residues of the function will be zero, if $R(t+x+y+z-2c) > -1$. The sum of the residues at the poles of $1/\sin\pi\zeta$ is S. For the poles of $\Pi(c-1-t+\zeta)$ the sum is, as in last article,

$$\pi\cot\pi(t-c)\sum_{n=0}^{n=\infty}(-c+2t-2n)\frac{(-1)^n}{\Pi n}\frac{1}{\Pi(2t-c-n)}\frac{\Pi(t-x-n-1)}{\Pi(t+x-c-n)} \text{ do. } y, z,$$

$$= -\cot\pi(t-c)\sin\pi(2t-c)\frac{\sin\pi(t+x-c)}{\sin\pi(t-x)} \text{ do. } y, z, \times$$

$$\sum_{n=0}^{n=\infty}(c-2t+2n)\frac{\Pi(c-1-2t+n)}{\Pi n}\frac{\Pi(c-1-t-x+n)}{\Pi(x-t+n)} \text{ do. } y, z.$$

The sum of the series in the last line is obtained from (32) by writing $c-2t$ for c, $x-t$ for x, etc., and is

$$\frac{\Pi(c-t-x-1)\,\Pi(c-t-y-1)\,\Pi(c-t-z-1)\,\Pi(t+x+y+z-2c)}{\Pi(y+z-c)\,\Pi(z+x-c)\,\Pi(x+y-c)}.$$

Hence (34) becomes

$$\pi^3\frac{\Pi(t+x+y+z-2c)}{\Pi(y+z-c)\,\Pi(z+x-c)\,\Pi(x+y-c)\,\Pi(t+x-c) \text{ do. } y, z}\times$$

$$\frac{\sin\pi(2t-c)\cos\pi(t-c)}{\sin\pi(t-c)\sin\pi(t-x) \text{ do. } y, z} \quad - \; - \; - \; - \; - \; - \quad (35)$$

The first line here is symmetrical in t, x, y, z; hence the sum of the residues of $f(\zeta)$ at the poles of the four Π functions $\Pi(c-1-t+\zeta)$, etc., (that is, $-$S), is the product of the first line of (35) by the sum of the four expressions like that in the second line.

To find the sum of these four expressions, consider the function of ζ,

$$\phi(\zeta) = \frac{\pi\sin\pi(2\zeta-c)\cos\pi(\zeta-c)}{\sin\pi(\zeta-c)\sin\pi(\zeta-t) \text{ do. } x, y, z},$$

within a strip of the ζ plane bounded by two lines parallel to the axis of imaginaries and at unit distance apart. On these two lines $\phi(\zeta)$ has equal values, and it vanishes exponential-wise at infinity. By Cauchy's Theorem the sum of the residues of $\phi(\zeta)$ at the poles within the strip accordingly vanishes.

But the second line of (35) is the residue at the pole of $1/\sin\pi(\zeta - t)$. Hence the sum of the four similar expressions

$$= -\text{residue at pole of } 1/\sin\pi(\zeta - c)$$

$$= -\frac{\sin\pi c}{\sin\pi(t-c)\text{ do. } x,\ y,\ z}.$$

Thus for the sum of the series in (33) we have

$$S = \frac{\pi^3 \sin\pi c}{\sin\pi(t-c)\text{ do. } x,\ y,\ z} \times$$

$$\frac{\Pi(t+x+y+z-2c)}{\Pi(y+z-c)\,\Pi(z+x-c)\,\Pi(x+y-c)\,\Pi(t+x-c)\text{ do. } y,\ z}, \quad (36)$$

$$\text{where } R(t+x+y+z-2c) > -1.$$

To exhibit the result as the summation of a series of rational terms, multiply both sides of (36) by

$$\frac{\Pi t\,\Pi x\,\Pi y\,\Pi z}{\Pi(c-1-t)\text{ do. } x,\ y,\ z}.$$

Then

$$c + (c+2)\,\frac{c-t}{t+1}\text{ do. } x,\ y,\ z,\ + \ldots + (c+2n)\,\frac{(c-t)^{(n)}}{(t+1)^{(n)}}\text{ do. } x,\ y\ z,\ + \ldots$$

$$+ (c-2)\,\frac{t}{c-t-1}\text{ do. } x,\ y,\ z,\ + \ldots + (c-2n)\,\frac{t^{(-n)}}{(c-t-1)^{(-n)}}\text{ do. } x,\ y,\ z,\ + \ldots$$

$$= \frac{\sin\pi c}{\pi}\,\frac{\Pi t\,\Pi x\,\Pi y\,\Pi z\,\Pi(t-c)\,\Pi(x-c)\,\Pi(y-c)\,\Pi(z-c)\,\Pi(t+x+y+z-2c)}{\Pi(y+z-c)\Pi(z+x-c)\Pi(x+y-c)\Pi(t+x-c)\Pi(t+y-c)\Pi(t+z-c)}. \quad (37)$$

For $t = 0$, this is equivalent to (9).

The result may be put in somewhat more striking form by writing $2a$ for c, and then $t+a$, $x+a$, $y+a$, $z+a$ for t, x, y, z.

Of special cases of (37), those obtained by writing $t = c/2$, $t = \infty$, $t = (c-1)/2$ may be mentioned.

On the Resolution of Integral Algebraic Expressions into Factors.

By R. F. Muirhead, M.A., D.Sc.

On Arithmetical Approximations.

By R. F. Davis, M.A.

Edinburgh Mathematical Society.

LIST OF MEMBERS.

TWENTY-FIFTH SESSION 1906–1907.

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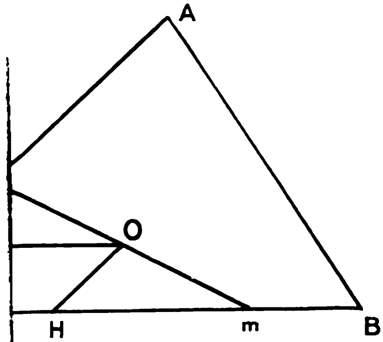

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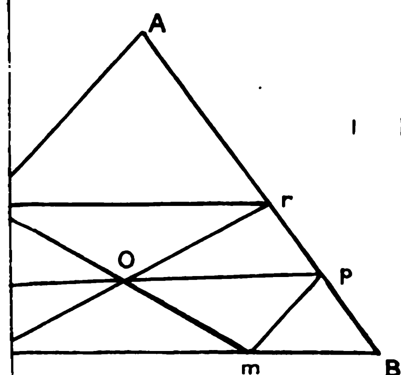

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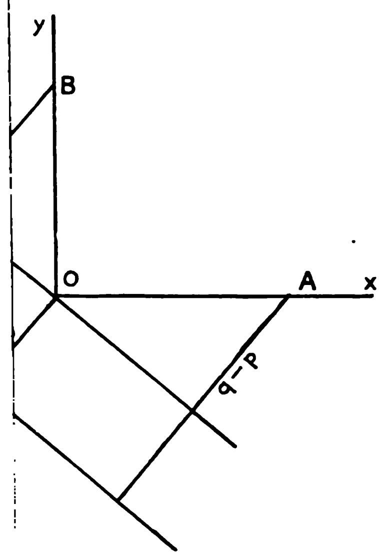

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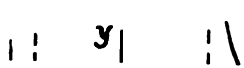

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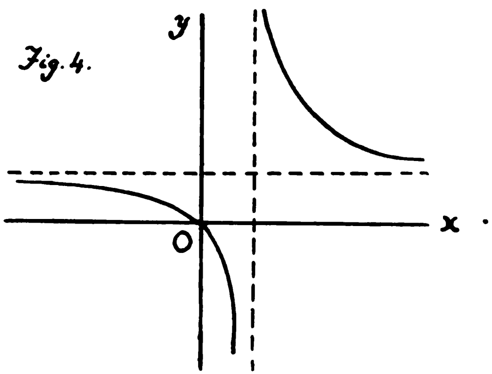

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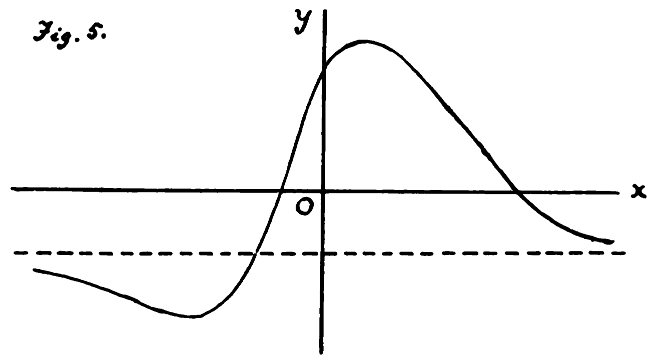

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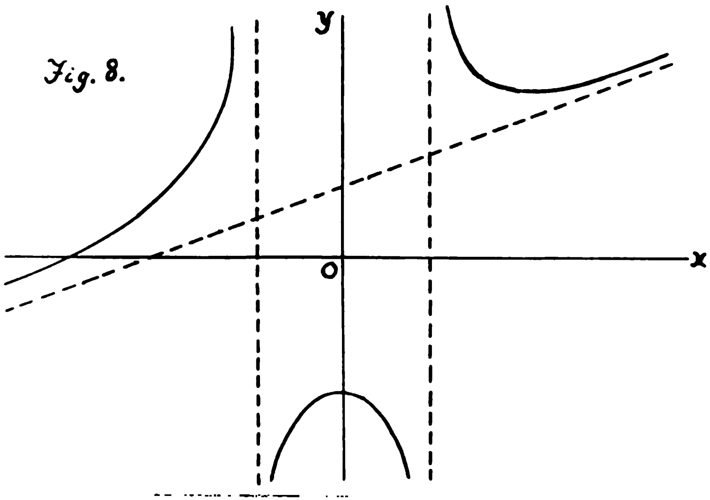

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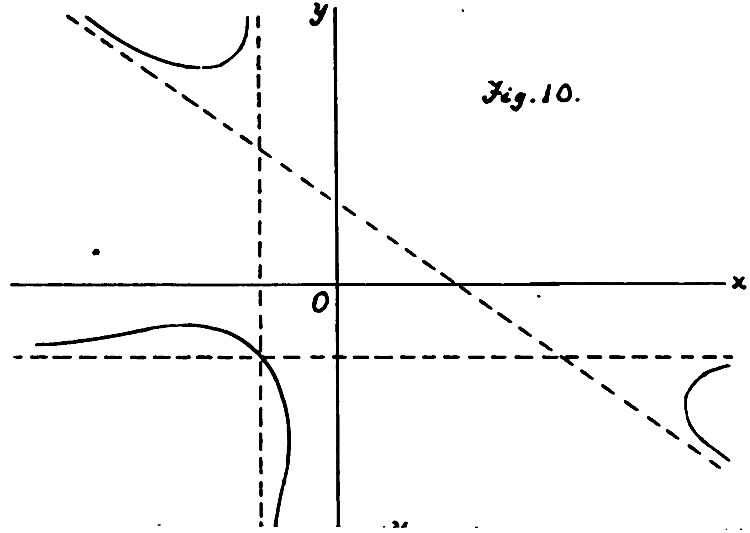

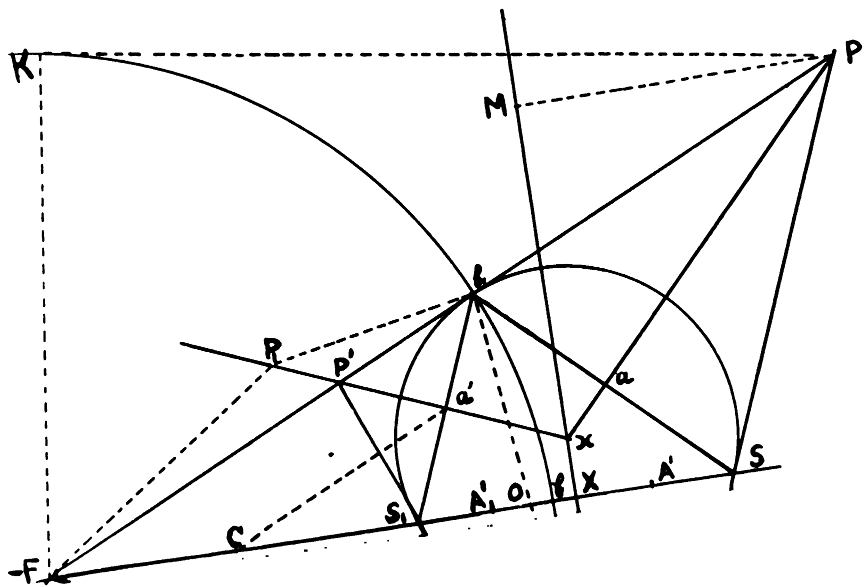
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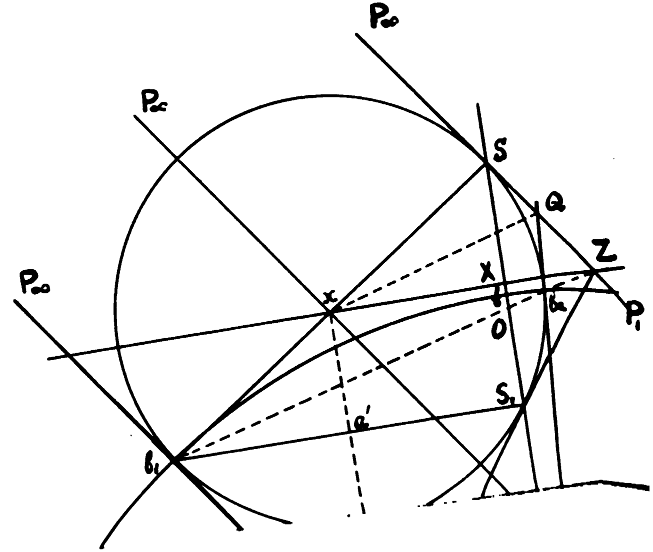
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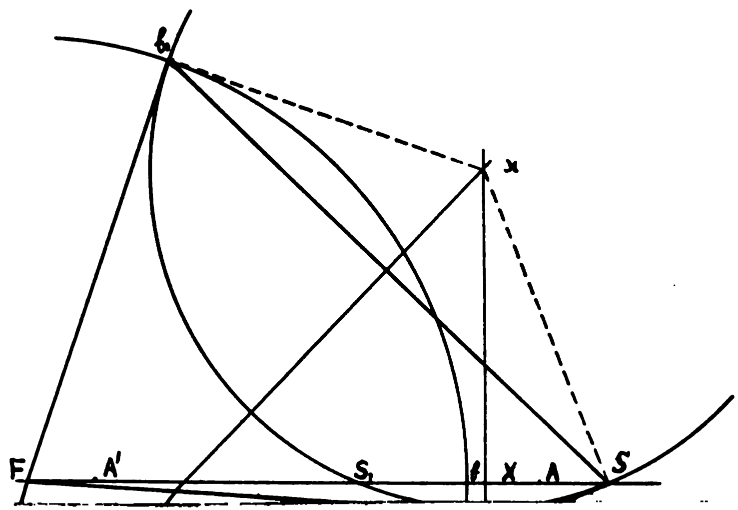

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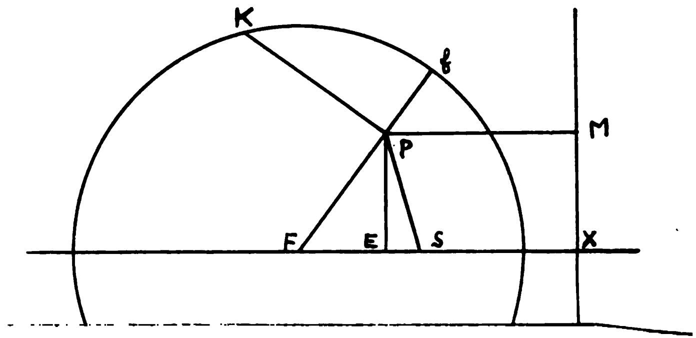

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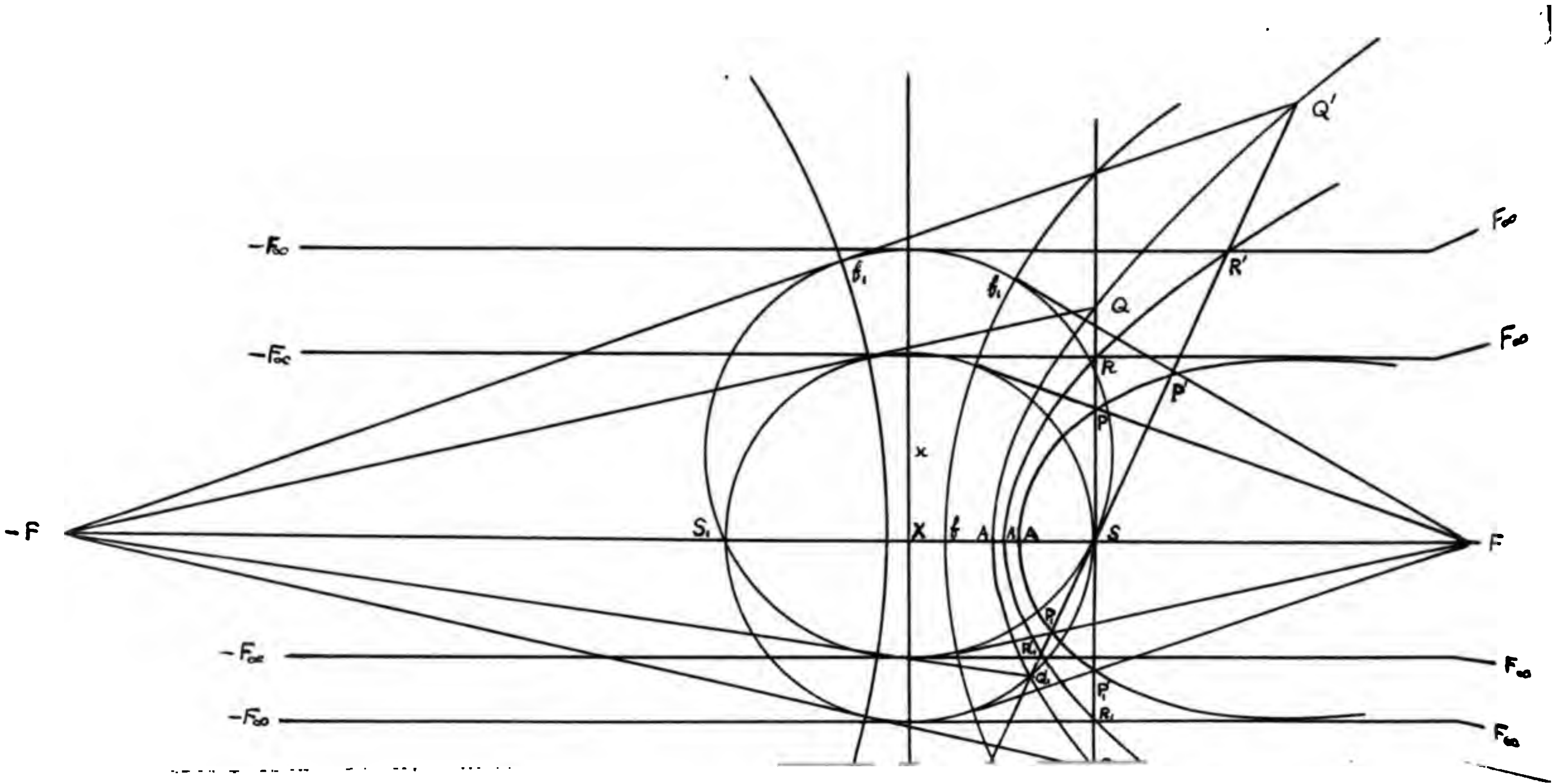

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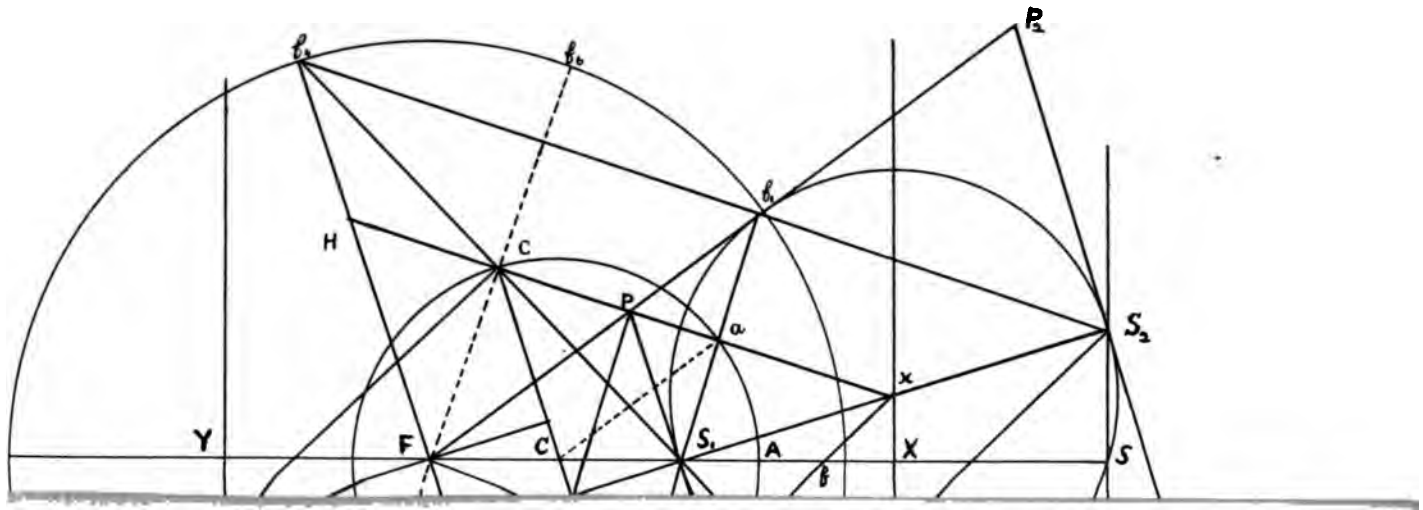

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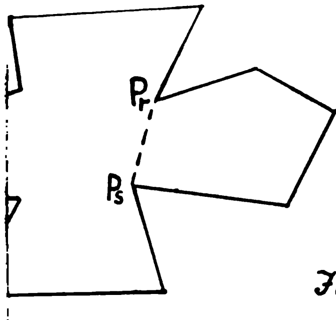

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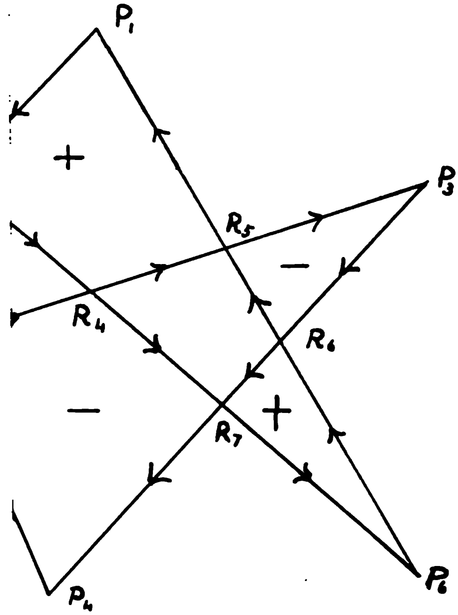

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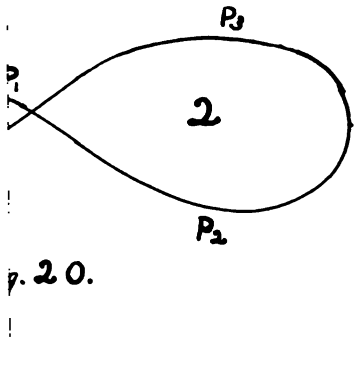

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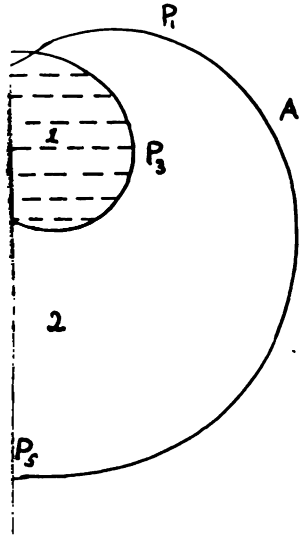

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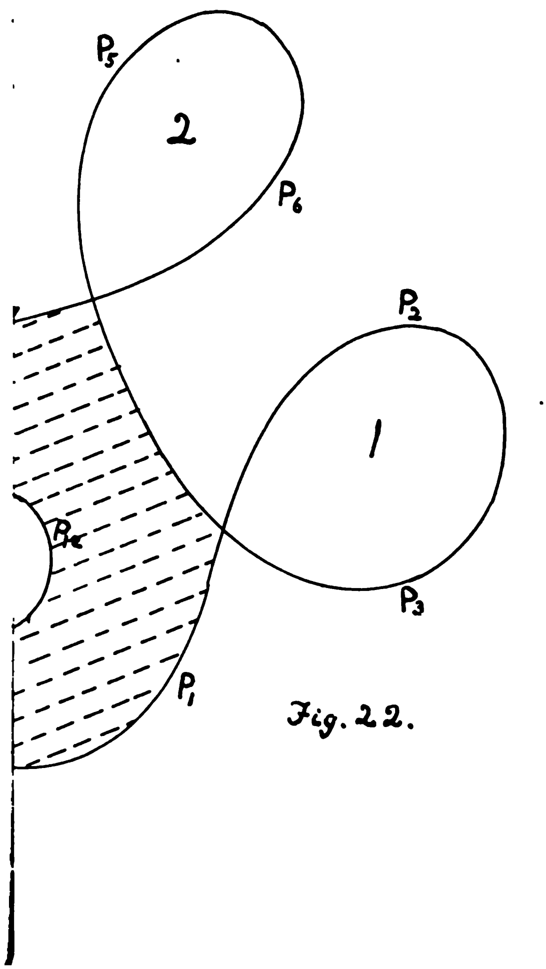

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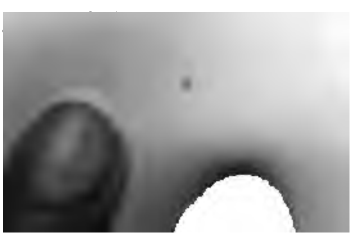


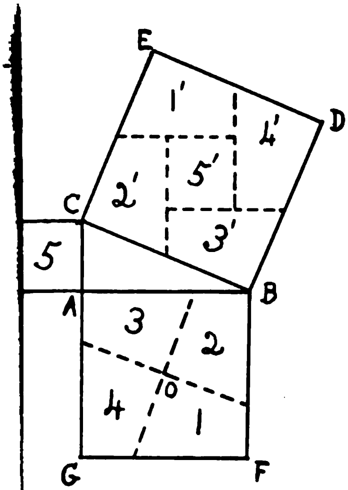

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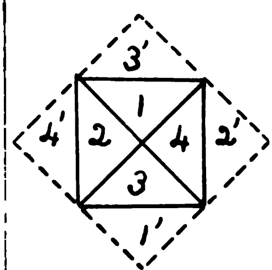

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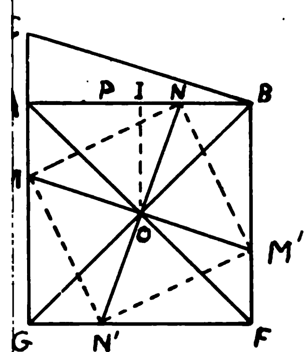

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TWENTY-SIXTH SESSION, 1907-1908.

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1. Poles and Polars of Conics - - -	Prof. J. JACK.
2. A Problem in Plane Geometry - - -	E. COLLIGNON.
3. A First Evaluation of π - - - -	R. F. DAVIS.

Second Meeting, Friday, 13th December, 1907.

1. The Dygogram of Axle Reaction of a Pendulum	Prof. GREENHILL.
2. Notes on Spherical Harmonics - - -	Dr DOUGALL.
3. On a Solution of the Apollonian Problem -	Dr DOUGALL.

Third Meeting, Friday, 10th January, 1908.

1. On the Lengths of a Pair of Tangents to a Conic	Prof. ANGLIN.
2. On the Lines which intersect Three Given Lines in Space - - - - - -	Prof. DIXON.
3. Proof that Every Rational Algebraic Equation has a Root - - - - -	Prof. DIXON.
4. Theorem regarding Orthogonal Conics - -	W. FINLAYSON.

Fourth Meeting, Friday, 14th February, 1908.

1. The Place of Graphs in a Course of Mathematics - - - - -	Dr C. McLEOD.

Fifth Meeting, Friday, 13th March, 1908.

1. On the Geometry of the Conic and Triangle -	JOHN MILLER.
2. Graphical Solution of n Simultaneous Linear Equations with n Unknowns. - -	Dr MUIRHEAD.
3. On Expressing a Determinant by Compound Determinants of the Second Order - -	Dr MUIRHEAD.
4. School Apparatus for Determining g - -	J. A. McBRIDE.

Sixth Meeting, Friday, 8th May, 1908.

1. Examples in the Geometry of Cross Ratio -	C. TWEEDIE.
2. The Intrinsic Properties of Curves - -	Prof. NARANIENGAR.
3. Geometrical Extraction of Cube Root - -	J. G. THOMSON.

Seventh Meeting, Friday, 12th June, 1908.

1. On Contact of Curves - - - -	Prof. NARANIENGAR.
2. On Factors of Numbers of the Form $\{x^{(2n+1)k} \pm 1\} \div \{x^k \pm 1\}$ - - -	Prof. SANJANA.
3. On a Problem in the Theory of Numbers -	Dr T. HUGH MILLER.

Corrections to a paper, "Coaxal Circles and Conics," by Wm. Finlayson, in Vol. XXV. of *Proceedings.*

Page 52, par. 9, line 2.

"$FP'F_1P$" should read "$FP'f_1P$."

Page 53, par. 14, line 11, insert after the word parabola

"indicated in Fig. 16 by $RR_1R'R_1'$ which also show that the parabola is the upper limit of the ellipse or lower limit of the hyperbola"

also for the word "and" substitute "then" in same line.

Page 54, par. 16, line 8, for "$f_1S_2f_2$" read "f_1xf_2," and in line 9, for "f_2Sf_4" read "$f_2S_2f_4$."

Page 55, par 19, line 3, for "f_1f_3" read "f_1f_3," and in line 4. for "ca'_1c_1" read "ca_1c_1" and in line 5, for "$xa_1Q_1C_1$" read "xa_1Q_1c"

in par. 20, same page, line 2, for "xS bisects $P'S_1Q'$" read "xS_1 bisects angle $P'S_1Q'$"

same par., line 8, for "direction" read "directrix."

Page 56, par. 24, line 1, for "CY and C_1Y" read "cy and c_1y," and in line 3, for "YF" read "yF," par. 25, line 5, for "$ScSx$" read "$SCSX$," and line 6, for "Sx" read "Sk," and for "circle" read "radius."

PROCEEDINGS

OF THE

EDINBURGH
MATHEMATICAL SOCIETY.

TWENTY-SIXTH SESSION, 1907-1908.

For this Session the following Office-Bearers were elected:—

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John Miller, M.A.

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Problem in Plane Geometry

By M. Edouard Collignon

Inspecteur général des Ponts et Chaussées en retraite, Examinateur honoraire à l'Ecole polytechnique, Paris.

(*Read November 8th, 1907. Received, same date.*)

Figure 1

In a plane, a point O and a straight line OH drawn through O are given. OH is the bisector of an unknown angle YOX, which it is required to determine by the following conditions:

A point I, given by position in the plane of the figure, is connected with the straight line OH by the given angle $IOH = \theta$, and by the distance $OI = c$, from the point I to the vertex of the angle. This point I is the middle of a chord AB inscribed in the angle YOX, Furthermore the product $OA \times OB$ of the distances to the point O of the extremities of this chord is equal to a given quantity K^2.

The data therefore are:

The bisector OH of the angle YOX or AOB given in direction, and the median OI, of the triangle AOB, given in direction and magnitude.

The polar co-ordinates $IOH = \theta$, $OI = c$ which fix the position of the point I with reference to the bisector.

Finally, the product $OA \times OB = K^2$ of the sides OA and OB of the triangle OAB.

The unknowns are:

The lengths $OA = x$, $OB = y$, $AB = 2z$ of the sides of the triangle.

The angle α which the sides OA and OB make with the bisector OH.

The angle ϕ which the median makes with the base AB.

The angles A and B of the triangle.

We shall begin by determining the angle α, which is the key of the solution. Once this angle is determined, the figure may be constructed.

Equations of the Problem

The angles AOI, BOI are expressed in terms of the angle α by the relations

$$AOI = \alpha + \theta$$
$$IOB = \alpha - \theta$$

Apply the well-known trigonometrical formula to the triangles IAO, BIO, BAO, as well as to the triangle OII′, which is obtained by drawing through the point I a parallel II′ to the side OB, till it meets the side OA in I′.

To place the chord AB, it will suffice to take on the axis OX a quantity $I'A = OI'$, then to join the point A to the point I We shall have

$$OI' = \frac{x}{2} \quad \text{and } I'I = \frac{y}{2}$$

The four triangles give us the equations

(1) $$z^2 = x^2 + c^2 - 2cx\cos(\alpha + \theta)$$

(2) $$z^2 = y^2 + c^2 - 2cy\cos(\alpha - \theta)$$

(3) $$4z^2 = x^2 + y^2 = 2xy\cos 2\alpha, \text{ and finally}$$

$$c^2 = \frac{x^2}{4} + \frac{y^2}{4} + 2\frac{x}{2} \times \frac{y}{2}\cos 2\alpha$$

changing here the sign of the cosine, because the angle 2α is exterior, and not interior, to the triangle considered.

The last equation may be written

(4) $$4c^2 = x^2 + y^2 + 2xy\cos 2\alpha$$

To these must be added the other condition expressed by

(5) $$xy = K^2$$

A combination of these five equations will give us the unknowns and specially the angle α.

SOLUTION OF EQUATIONS (1)—(5)

In the equations (3) and (4) substitute K^2 for the product xy; then

(3′) $$4z^2 = x^2 + y^2 - 2K^2\cos 2\alpha$$

(4′) $$4c^2 = x^2 + y^2 + 2K^2\cos 2\alpha$$

By addition and subtraction we obtain

(3″) $$z^2 + c^2 = \tfrac{1}{2}(x^2 + y^2)$$

(4″) $$z^2 - c^2 = -K^2\cos 2\alpha$$

The first of these equations expresses the well-known theorem which connects the median with the three sides of a triangle. From the second equation we have

(6) $$z^2 = c^2 - K^2 \cdot \cos 2\alpha.$$

Substitute this value of z^2 in equation (3″) and we obtain

$$(7) \qquad x^2 + y^2 = 4c^2 - 2K^2 \cos 2\alpha.$$

If we substitute these values of z^2 and $x^2 + y^2$ in equations (1) and (2), there results

$$\begin{aligned} c^2 - K^2 \cos 2\alpha &= x^2 + c^2 - 2cx \cos(\alpha + \theta) \\ &= y^2 + c^2 - 2cy \cos(\alpha - \theta). \end{aligned}$$

Solving these two equations with respect to $\cos(\alpha + \theta)$ and $\cos(\alpha - \theta)$, we have

$$(8) \qquad \begin{cases} \cos(\alpha + \theta) = \dfrac{x^2 + K^2 \cos 2\alpha}{2cx} \\ \cos(\alpha - \theta) = \dfrac{y^2 + K^2 \cos 2\alpha}{2cy} \end{cases}$$

The final equation which will give us $\cos\alpha$ is obtained by multiplication, from equations (8). It is

$$(9) \quad \cos(\alpha + \theta)\cos(\alpha - \theta) = \frac{(x^2 + K^2 \cos 2\alpha)(y^2 + K^2 \cos 2\alpha)}{2cx \times 2cy}$$

$$= \frac{x^2 y^2 + K^2 \cos 2\alpha \, (x^2 + y^2) + K^4 \cos^2 2\alpha}{4c^2 \, xy}$$

and this lends itself to important simplifications.

1°. The first member simplifies to $\cos^2\alpha - \sin^2\theta$.

2°. The second member contains only the product xy, its square x^2y^2, and the sum $x^2 + y^2$, all of them functions which may be expressed by means of $\cos\alpha$, from the relations (5) and (7).

We have the identity

$$\begin{aligned} & x^2y^2 + K^2 \cos 2\alpha \, (x^2 + y^2) + K^4 \cos^2 2\alpha \\ &= K^4 + K^2 \cos 2\alpha \, (4c^2 - 2K^2 \cos 2\alpha) + K^4 \cos^2 2\alpha \\ &= K^2(K^2 + 4c^2 \cos 2\alpha - 2K^2 . \cos^2 2\alpha + K^2 \cos^2 2\alpha) \\ &= K^2(K^2 + 4c^2 \cos 2\alpha - K^2 \cos^2 2\alpha). \end{aligned}$$

Hence by these transformations, and by suppressing, in numerator and denominator, the factor K^2, equation (9) becomes

$$(10) \qquad \cos^2\alpha - \sin^2\theta = \frac{K^2 + 4c^2 \cos 2\alpha - K^2 \cos^2 2\alpha}{4c}$$

Equation (10) is a biquadratic in $\cos\alpha$, that is, an equation of the second degree in $\cos^2\alpha$.

Put therefore $\cos^2 a = u$, calling u a new unknown. We have then

$$\cos 2a = 2u - 1, \; \cos^2 2a = 4u^2 - 4u + 1$$

and equation (10) takes the form

$$u - \sin^2\theta = \frac{K^2}{4c^2} + 2u - 1 - \frac{K^2}{4c^2}(4u^2 - 4u + 1)$$

Arrange with respect to u, and divide by $\frac{K^2}{c^2}$, then

$$(11) \qquad u^2 - \left(1 + \frac{c^2}{K^2}\right)u + \frac{c^2}{K^2}\cos^2\theta = 0.$$

The two values of u are

$$(12) \qquad u = \frac{1}{2} + \frac{c^2}{2K^2} \pm \sqrt{\frac{1}{4} + \frac{c^2}{2K^2} + \frac{c^4}{4K^4} - \frac{c^2}{K^2}\cos^2\theta}$$

In order that the roots may be real, we must have the inequality

$$\frac{c^4}{4K^4} + \frac{c^2}{2K^2} + \frac{1}{4} - \frac{c^2}{K^2}\cos^2\theta > 0$$

Now, identically

$$\frac{c^2}{2K^2} - \frac{c^2}{K^2}\cos^2\theta = \frac{c^2}{2K^2}(1 - 2\cos^2\theta)$$

$$= -\frac{c^2}{2K^2}\cos 2\theta$$

The inequality, therefore, after clearing of fractions, becomes

$$c^4 + K^4 - 2c^2K^2\cos 2\theta > 0$$

and we see that this condition will always be fulfilled.

Imagine a triangle A′O′B′ whose sides are proportional respectively to the squares c^2 and K^2 and which contain between them the angle 2θ.

The first member will be the square of the third side A′B′ homogeneous to the square of a length, and consequently always positive.

One could see the same thing otherwise by noticing that $c^4 + K^4 - 2c^2K^2$ is the square, necessarily positive, of $c^2 - K^2$; and that the introduction of the factor $\cos 2\theta$, numerically inferior to unity, into the negative term cannot modify this result.

The equation (12) therefore assigns to the unknown quantity u real and consequently positive values. To each one of these values of u correspond two values of $\cos\alpha$,

$$\cos\alpha = \pm\sqrt{u}$$

But the negative root is inadmissible, for $\cos\alpha$ negative defines an angle α comprised between $\frac{\pi}{2}$ and π, and would make of the angle 2α, which is that of the triangle BOA, an angle greater than π. The value π, attributed to the angle 2α, is a limiting value which it is impossible to attain; the chord AB drawn through a given point I could not be inserted in an angle equal to two right angles, whose sides would be the prolongation of each other.

There are therefore two solutions to the problem, since two values of u, real and positive, satisfy equation (12); but there are not four, since the negative determinations of $\cos\alpha$ are wanting.

Sequel to the Solution

When the values of $\cos\alpha$ have been found, Figure 1 can be constructed by drawing the corresponding lines OX, OY, and this completes the solution graphically.

The calculation of x, y, z conducts likewise to the complete solution of the problem.

The value of z is obtained by the equation

(6) $z^2 = c^2 - K^2\cos 2\alpha$; then those of x and y by the solution of equations (8)

$$(8) \qquad \begin{cases} x^2 - 2cx\cos(\alpha+\theta) + K^2\cos 2\alpha = 0 \\ y^2 - 2cy\cos(\alpha-\theta) + K^2\cos 2\alpha = 0 \end{cases}$$

Hence

$$z = \sqrt{c^2 - K^2\cos 2\alpha}$$
$$x = c\cos(\alpha+\theta) \pm \sqrt{c^2\cos^2(\alpha+\theta) - K^2\cos 2\alpha}$$
$$y = c\cos(\alpha-\theta) \pm \sqrt{c^2\cos^2(\alpha-\theta) - K^2\cos 2\alpha}$$

The positive value of z alone is kept. As to x and y the two values which the formulae give are all positive if they are real, and that takes for granted the inequality

$$\frac{c^2}{K^2} > \frac{\cos 2\alpha}{\cos^2(\alpha+\theta)}$$

But these two values are not admissible without examination; the unknowns x and y are bound to verify the equation

(5) $xy = K^2$

One can therefore admit only those simultaneous values of x and y which satisfy this condition. Any combination taken arbitrarily among the values of x and y will not verify equation (5).

Let x', x'' be the two values of x
y', y'' " " " " " y

We shall have, by equations (8)

$$(a) \qquad \begin{cases} x'\ x'' = K^2 \cos 2\alpha \\ y'\ y'' = K^2 \cos 2\alpha \end{cases}$$

Suppose that the combination $(x'\ y')$ agrees with equation (5) so that

$$(b) \qquad x'\ y' = K^2$$

From the two equations (a) by multiplication we have

$$x'\ y'\ x''\ y'' = K^4 \cos^2 2\alpha\,;$$

and since, by hypothesis, $x'\ y' = K^2$

it follows that $x''\ y'' = K^2 \cos^2 2\alpha$

This shows that the combination (x'', y'') does not satisfy the imposed condition (5), except in the case when $2\alpha = 0$, and this case may be put aside to begin with, since it annuls the angle AOB.

Only those roots of equations (8) should be associated together which verify equation (5).

Knowing the sides of the triangles BOI, IOA, the angles of these triangles may be deduced either by the proportion of the sines, or by any other geometrical means.

Figure 2

If, for example, A′ and B′ be the projections of A and B on OI, then

$$A'I = B'I = c - x\cos(\alpha + \theta)$$
$$= y\cos(\alpha - \theta) - c$$

whence $$\cos\phi = \frac{A'I}{AI} = \frac{c - x\cos(\alpha+\theta)}{z}$$

$$\text{angle IAO} = \pi - (\alpha + \theta + \phi)$$
$$\text{angle IBO} = \phi - \alpha + \theta$$

The area of triangle AOB has for measure

$$\frac{1}{2}xy\sin 2\alpha = \frac{K^2\sin 2\alpha}{2}$$

The altitude OS drawn from O to the base AB of triangle AOB is given by the relation

$$OS = \frac{K^2\sin 2\alpha}{4z}$$

It is also equal to $\quad x\sin A = x\sin(\alpha+\theta+\phi)$
or to $\quad y\sin B = y\sin(\phi-\alpha+\theta)$

whence angle ϕ may be determined if x, y, α, and θ are known.

Particular Case of the Right-Angled Triangle

Figure 3

If angle 2α is right we have

$$\alpha = \frac{\pi}{4}, \qquad u = \cos^2\alpha = \frac{1}{2}$$

The results obtained are verified directly on the figure.

For
$$x = OA = 2c\cos(\alpha+\theta) = 2c\cos A$$
$$y = OB = 2c\cos(\alpha-\theta) = 2c\sin A;$$

angles $\alpha+\theta$, $\alpha-\theta$ are complementary

$$z = IB = IA = OI = c$$

Furthermore
$$\begin{aligned} K^2 = xy = 4c^2\sin A\cos A &= 2c^2\sin 2A \\ &= 2c^2\sin 2(\alpha+\theta) \\ &= 2c^2\sin\left(\frac{\pi}{2}+2\theta\right) \\ &= 2c^2\cos 2\theta \end{aligned}$$

The area of triangle AOB is equal to $\frac{1}{2}K^2$

Particular Case of the Isosceles Triangle

In the isosceles triangle defined by the condition $x=y$ we have $\theta=0$, for the mid point I of the base AB is situated on the bisector of angle AOB. We have then

$$x = y = K$$

and equation (11) can be decomposed into two factors

$$u^2 - \left(1+\frac{c^2}{K^2}\right)u + \frac{c^2}{K^2} = (u-1)\left(u-\frac{c^2}{K^2}\right) = 0$$

Of the two roots $u=1$ and $u=\frac{c^2}{K^2}$, the first may be set aside, since it involves $\alpha=0$ or $\alpha=\pi$; the second gives $c=K\cos\alpha$, a relation readily verified from a figure, as well as the relation $z=c\tan\alpha$.

General Remark

As soon as the angle 2α is determined, if we make the angle θ vary from the value $\theta=0$ to the values $-\alpha$ and $+\alpha$, the straight line AB varies, and envelops a hyperbola which has O for centre and OX, OY for asymptotes; the moveable straight line touches its envelope at its mid point I. The angle ϕ is then the angle formed by the tangent to the curve and the radius vector OI.

If we call r the radius vector expressed in function of the angle θ taken as polar angle, we shall have

$$\tan\phi=\frac{r}{r'}=\frac{rd\theta}{dr}$$

The equation of the curve is

$$r=\frac{a\sin\alpha}{\sqrt{\cos^2\theta-\cos^2\alpha}}$$

Hence, by applying the formula,

$$\tan\phi=\frac{\cos^2\theta-\cos^2\alpha}{\cos\theta\sin\theta}$$

The factor a is the semi-axis of the curve, measured on the bisector OH of the angle of the asymptotes. At this point $\phi=\frac{\pi}{2}$. The asymptotes are given by the equality $\theta=\pm\alpha$.

The quantity a will be expressed in function of the data c and θ_0 by calling θ_0 the particular value of the polar angle furnished by the data of the problem.

Hence $$c=\frac{a\sin\alpha}{\sqrt{\cos^2\theta_0-\cos^2\alpha}}$$

which involves for a the value

$$a=\frac{c\sqrt{\cos^2\theta_0-\cos^2\alpha}}{\sin\alpha}$$

It can be seen that the problem is impossible if $\sin\alpha=0$, for then the semi-axis a of the hyperbola becomes infinite.

On the lines which intersect three given lines in space

By Professor A. C. Dixon

(*Read* 10*th January*, 1908. *Received, same date.*)

The following is an elementary discussion of certain propositions in solid geometry which are commonly left to a later stage.

Let ABCD be a quadrilateral formed of rods freely jointed, let points E, F, G, H be taken on AB, BC, CD, DA respectively and let EG, FH be joined by rods freely jointed to the former. Let us investigate whether the resulting framework is rigid.

It will be admitted that, when the given conditions enable us to find the lengths AC, BD, they fix the form of the framework. Let AE, EB, BF, FC, CG, GD, DH, HA, EG, FH be denoted by a, b, c, d, e, f, g, h, x, y respectively.

Then we have

$$b.\mathrm{CA}^2 + a.\mathrm{CB}^2 - (a+b)\,\mathrm{CE}^2 = ab(a+b),$$
$$f.\,\mathrm{EC}^2 + e.\mathrm{ED}^2 - (e+f)\,\mathrm{EG}^2 = ef(e+f),$$
$$b.\mathrm{DA}^2 + a.\mathrm{DB}^2 - (a+b)\mathrm{DE}^2 = ab(a+b)$$

and by elimination of CE, DE

$$fb.\mathrm{CA}^2 + ea.\mathrm{DB}^2 = (x^2+ab+ef)\,(a+b)\,(e+f) - af(c+d)^2 - be(g+h)^2 \quad - \quad (1)$$

for $\mathrm{BC} = c+d$, $\mathrm{DA} = g+h$.

In like manner, or by the interchanges ad, bc, eh, fg, xy, we have

$$cg.\mathrm{CA}^2 + dh.\mathrm{DB}^2 = (y^2+cd+gh)(c+d)(g+h) - dg(a+b)^2 - ch(e+f)^2 \quad - \quad (2)$$

The equations (1) and (2) determine CA^2 and DB^2 unless $aceg = bdfh$, that is, unless EFGH lie in one plane or in other words EG meets FH. Hence *the framework is rigid unless the rods* EG, FH *meet each other.*

If they meet then AC, BD are not determined by (1) and (2) but a relation must be satisfied by x, y, a, b...that is, the connexion FH suffices to keep the distance EG constant, and *vice versa.*

Further, if EG, FH meet in P, we have the sides of the quadrilateral ABFH met by a plane in E, C, P, D and therefore

$$\frac{\mathrm{HP}}{\mathrm{PF}} = \frac{\mathrm{HD}}{\mathrm{DA}} \cdot \frac{\mathrm{AE}}{\mathrm{EB}} \cdot \frac{\mathrm{BC}}{\mathrm{CF}} = \frac{g}{g+h} \cdot \frac{a}{b} \cdot \frac{c+d}{d}.$$

Hence P is a fixed point in FH and similarly in EG. Thus there would be no gain of rigidity by freely jointing the rods EG, FH to each other at P.

There would be no loss of rigidity if EG were taken away, nor on the other hand would there be any gain if another rod were added, connecting a point on AB with one on CD and *meeting FH*, or even any number of such rods, even if freely jointed with FH. The same is true of any number of rods from AD to BC, meeting EG.

Suppose then that we add E′G′ a rod meeting AB, FH, CD and F′H′ a rod meeting AD, EG, BC: the framework is not made stiff, and still less would E′G′ and F′H′ by themselves stiffen the quadrilateral ABCD. Hence E′G′ and F′H′ must meet each other. It follows then that *if three lines all meet three others, any line meeting the first triad and any line meeting the second triad must meet each other, and if any number of such lines are replaced by freely jointed rods the resulting framework is deformable.*

The first part of this theorem could be proved separately as follows, with the former notation.

Since EG meets FH, E, F, G, H lie in a plane and

$$\mathrm{AE}.\mathrm{BF}.\mathrm{CG}.\mathrm{DH}=\mathrm{EB}.\mathrm{FC}.\mathrm{GD}.\mathrm{HA}\,; \qquad (3)$$

Since E′G′ meets FH, similarly

$$\mathrm{AE}'.\mathrm{BF}.\mathrm{CG}'.\mathrm{DH}=\mathrm{E}'\mathrm{B}.\mathrm{FC}.\mathrm{G}'\mathrm{D}.\mathrm{HA}\,; \qquad (4)$$

Since EG meets F′H′,

$$\mathrm{AE}.\mathrm{BF}'.\mathrm{CG}.\mathrm{DH}'=\mathrm{EB}.\mathrm{F}'\mathrm{C}.\mathrm{GD}.\mathrm{H}'\mathrm{A} \qquad (5)$$

Multiplying (4) and (5) and dividing by (3) we have

$$\mathrm{AE}'.\mathrm{BF}'.\mathrm{CG}'.\mathrm{DH}'=\mathrm{E}'\mathrm{B}.\mathrm{F}'\mathrm{C}.\mathrm{G}'\mathrm{D}.\mathrm{H}'\mathrm{A}$$

from which it follows that E′G′ meets F′H′.

We see also that the quadrilateral ABCD will not be stiffened by two rods, EG, E′G′ freely jointed with AB and CD if EG, E′G′, AD, BC all meet one line. It is easy to prove that this is the only case of failure.

The equation (1) still holds, and if AE′, E′B, CG′, G′D, E′G′ are denoted by a', b', e', f', x' respectively, we have also

$$f'b'.\mathrm{CA}^2+e'a'.\mathrm{DB}^2=(x'^2+a'b'+e'f')(a'+b')(e'+f') \\ -a'f'(c+d)^2-b'e'(g+h)^2 \qquad (6)$$

The equations (1) (6) fix CA, DB, so that the framework is stiff, unless $fb/f'b'=ea/e'a'$.

This condition expresses that the cross ratios AEE′B, DGG′C are equal and also shews that any line meeting AD, BC, EG, will meet E′G′,

These results can be roughly verified by using hat-pins jointed together with small pieces of sheet india-rubber, as, for instance, from an old bicycle tire.

It has been assumed that the quadrilateral ABCD will not be stiffened by a single rod such as EG or FH. To see this, take ABCD with AB and AD fixed at any angle: then BD is fixed and the locus of G is a circle with BD as axis: this circle will meet a sphere, with centre E and radius not too small or too great, in two points. Thus EG can take any value, within certain limits, when the angle BAD has any value, or in other words fixing the length EG does not fix the angle BAD in general. There is an exception when EG has its maximum or minimum value.

Proof that every rational algebraic equation has a root

By Professor A. C. Dixon

(*Read* 10*th January*, 1908. *Received, same date.*)

The following arrangement of the proof of this theorem could, I think, be given at a comparatively early stage, even if the necessary case of De Moivre's theorem had to be proved as an introductory lemma.

Let u, v be two rational integral algebraic functions of x, y with real coefficients, and let c be a simple closed contour in the plane. As the point (x, y) travels round c let those changes in the sign of u that take place when v is positive be marked and let $(u, v\,;\,c)$ denote the excess in number among these of changes from + to − over changes from − to + *.

If c is deformed continuously, $(u, v\,;\,c)$ will not be changed except (1), when c passes over a point where u, v both vanish, (2), when there is a change in the number of points where c meets one of the curves $u=0$, $v=0$. In case (1) there will generally be a change in the value of $(u, v\,;\,c)$ since a change of sign in u on c will pass from the part of c where v is negative to that where v is positive, or conversely.

In case (2) suppose c to be deformed so that the number of its intersections with the curve $u=0$ is increased. The increase must be an even number since both curves are continuous and endless. The sign of v is constant in the neighbourhood unless we are dealing with a case under (1) and since the new changes in sign of u are alternately + − and − + there is no effect on $(u, v\,;\,c)$: similarly if the number of intersections with $u=0$ is decreased.

If the number of intersections of c with the curve $v=0$ is altered, it must again be by an even number and there will be no change in $(u, v\,;\,c)$ unless $u=0$ at the same place as v when the case falls under (1).

Hence the deformation of c produces no effect on $(u, v\,;\,c)$ unless c passes over a point where $u=0=v$. In particular, if c does not contain such a point, it can be made to shrink up to a small contour in a neighbourhood where u, v are of constant signs and $(u, v\,;\,c)$ being unaffected by this process must be 0 throughout.

* It is not hard to see that $(v,u\,;\,c)=(-u,v\,;\,c)=-(u,v\,;\,c)$ etc.

Now take u, v to be given by the equation $f(z) = u + \iota v$ where $z = x + \iota y$ and $f(z) = z^n + (a_1 + \iota b_1) z^{n-1} + (a_2 + \iota b_2) z^{n-2} + \ldots$: let c be a circle with centre at the origin and radius R, so that on c $z = \mathrm{R}\ (\cos\theta + \iota\sin\theta)$, and θ runs from 0 to 2π.

We have by De Moivre's theorem

$$u = \mathrm{R}^n \cos n\theta + \mathrm{R}^{n-1}(a_1 \cos \overline{n-1}\theta - b_1 \sin \overline{n-1}\theta) + \text{lower powers of R},$$
$$v = \mathrm{R}^n \sin n\theta + \mathrm{R}^{n-1}(a_1 \sin \overline{n-1}\theta + b_1 \cos \overline{n-1}\theta) + \text{lower powers of R}.$$

The sum of all the terms after the first, either in u or v, is not greater than the sum of $a_1\mathrm{R}^{n-1}$, $b_1\mathrm{R}^{n-1}$, $a_2\mathrm{R}^{n-2}$... all taken positively and a value of R may be chosen so great that this sum does not exceed $k\mathrm{R}^n$ where k is any finite quantity. Thus the sign of u will be that of its first term if $\cos^2 n\theta > k^2$ and similarly for v: we shall take $k = \frac{1}{2}$.

Divide c into $4n$ parts at the points where

$$\theta = (2r+1)\,\pi/4n \qquad (r = 0, 1, \ldots 4n-1)$$

In the part of c when θ rises from $(8m-1)\pi/4n$ to $(8m+1)\pi/4n$ $\cos n\theta \not< 1/\sqrt{2}$ and thus u is positive while the sign of v is at first $-$ and at last $+$.

When θ rises from $(8m+1)\pi/4n$ to $(8m+3)\pi/4n$, v is positive, but u begins $+$ and ends $-$.

When θ rises from $(8m+3)\pi/4n$ to $(8m+5)\pi/4n$, u is always $-$.

When θ rises from $(8m+5)\pi/4n$ to $(8m+7)\pi/4n$, v is always $-$.

Only in the second case is there any contribution to $(u, v\,;\,c)$ and as this case occurs n times contributing 1 each time we have

$$(u, v\,;\,c) = n.$$

Hence c must contain a point where u, v vanish together, and the equation $f(z) = 0$ must have a root.

The proof that there are exactly n roots is now easy.

To express a Determinant of the *n*th Order in terms of Compound Determinants of the 2nd Order, and *vice-versa.*

By R. F. Muirhead.

(*Read 13th March. Received same date.*)

1. Let $\phi\ (a\, b'\, c'')$ denote the compound determinant,

$$\begin{vmatrix} (b'\, c''),\ (a'\, c'') \\ (b\ \ c''),\ (a\ \ c'') \end{vmatrix},\ \text{where } (b'\, c'') \text{ denotes} \begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix} \text{ etc.}$$

Then if A, B, etc., denote the co-factors of the elements a, b, etc. in the determinant $(a\, b'\, c'')$, we have

$$\phi\ (a\, b'\, c'') = \begin{vmatrix} \mathrm{A} & -\mathrm{B} \\ -\mathrm{A}' & \mathrm{B}' \end{vmatrix} = c''\ (a\, b'\, c'').$$

2. Again, denoting by $\phi\ (a\, b'\, c''\, d''')$ the compound determinant

$$\begin{vmatrix} \phi\ (b'\, c''\, d'''),\ \phi\ (a'\, c''\, d''') \\ \phi\ (b\ \ c''\, d'''),\ \phi\ (a\ \ c''\, a''') \end{vmatrix},\ \text{we have}$$

$$\phi\ (a\, b'\, c''\, d''') = \begin{vmatrix} d'''\ (b'\, c''\, d'''),\ d'''\ (a'\, c''\, d''') \\ d'''\ (b\ \ c''\, d'''),\ d'''\ (a\ \ c''\, d''') \end{vmatrix} = d'''^{2} \begin{vmatrix} \mathrm{A} & -\mathrm{B} \\ -\mathrm{A}' & \mathrm{B}' \end{vmatrix}$$

$$= d'''^{2} . (c''\, d''') . (a\, b'\, c''\, d''').$$

Here A, B, etc., are the cofactors of a, b, etc., in the determinant $(a\, b'\, c''\, d''')$.

3. The general formula of which the two preceding are special cases is

$$\phi\ (a_1\, b_2\, c_3 \ldots\ldots t_n) = (t_n)^{2^{n-3}} . (s_{n-1} t_n)^{2^{n-4}} \ldots\ldots (c_3\, d_4 \ldots\ldots t_n) . (a_1\, b_2\, c_3 \ldots\ldots t_n)$$

which can be established by mathematical induction without difficulty, observing that

$$\phi\ (a_1\, b_2\, c_3 \ldots\ldots t_n u_{n+1}) \equiv \begin{vmatrix} \phi\ (b_2\, c_3 \ldots\ldots u_{n+1}),\ \phi\ (a_2\, c_3 \ldots\ldots u_{n+1}) \\ \phi\ (b_1\, c_3 \ldots\ldots u_{n+1}),\ \phi\ (a_1\, c_3 \ldots\ldots u_{n+1}) \end{vmatrix}$$

and using the well-known theorem that in the determinant $(a_1\, b_2 \ldots t_n)$

$$\begin{vmatrix} \mathrm{A}_1 & \mathrm{B}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 \end{vmatrix} = (c_3\, d_3 \ldots\ldots t_n) . (a_1\, b_2\, c_3 \ldots\ldots t_n)$$

4. Now let us denote $\phi(a_1 b_2 c_3 \ldots\ldots l_n)$ by ϕ_1

$\phi(b_2 c_3 \ \ldots\ldots l_n)$ by ϕ_2

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\phi(r_{n-2} s_{n-1} t_n)$ by ϕ_{n-2}

$\phi(s_{n-1} t_n)$ by ϕ_{n-1}

Also denote $(a_1 b_2 \ldots\ldots l_n)$ by Δ_1

$(b_2 c_3 \ \ldots l_n)$ by Δ_2

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$(s_{n-1} t_n)$ by Δ_{n-1}

t_n by Δ_n or α_n

so that $\alpha_{n-1} = \Delta_{n-1} = \begin{vmatrix} s_{n-1} & t_{n-1} \\ s_n & t_n \end{vmatrix}$

The formula of the preceding article can now be written

$$\phi_1 = \Delta_1^{2^{n-3}} \cdot \Delta_2^{2^{n-4}} \cdot \Delta_3^{2^{n-5}} \ldots\ldots\ldots\ldots\ldots\ldots \Delta_{n-3}^{2^2} \cdot \Delta_{n-2}^{2} \cdot \Delta_{n-1} \Delta_{n-1}^{0} \Delta_n$$

Hence also

$$\phi_2 = \Delta_2^{2^{n-4}} \cdot \Delta_3^{2^{n-5}} \cdot \Delta_4^{2^{n-6}} \ldots\ldots\ldots\ldots\ldots\ldots \Delta_{n-2} \Delta_{n-1}^{1} \Delta_n$$

$$\phi_3 = \Delta_3^{2^{n-5}} \cdot \Delta_4^{2^{n-6}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Delta_{n-2} \Delta_{n-1}^{1} \Delta_n$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots \ . \ \ldots\ldots\ldots\ldots$

$\phi_{n-3} = \Delta_{n-3}^{2} \Delta_{n-2} \Delta_{n-1}^{1} \Delta_n$

$\phi_{n-2} = \Delta_{n-2} \Delta_{n-1}^{1} \Delta_n$

$\phi_{n-1} = \Delta_{n-1}^{1} \Delta_n$

$\phi_n = \Delta_n$

5. Hence we deduce

$$\alpha_1 \alpha_2^{2} \alpha_3^{2^2} \ldots\ldots \alpha_{n-1}^{2^{n-2}} \alpha_n^{2^{n-2}} = \Delta_1 \cdot \Delta_2^{2} \cdot \Delta_3^{2^2} \ldots\ldots \Delta_{n-1}^{2^{n-2}} \cdot \Delta_n^{2^{n-2}}.$$

To prove this we have to show that the exponent of Δ_{n-r} in the product, viz.,

$2^{n-r-3} + 2 \cdot 2^{n-r-4} + 3 \cdot 2^{n-r-5} + \ldots + \overline{(n-r-3)} \cdot 2 + (n-r-4) + (n-r-2)$ is $= 2^{n-r-1}$.

Now the former expression may be written

$$
\begin{array}{l}
2^{n-r-5}+2^{n-r-6}+2^{n-r-7}+\dots\dots+2^2+2^1+1+1 \\
\quad +2^{n-r-6}+2^{n-r-7}+\dots\dots+2^2+2^1+1+1 \\
\qquad +\dots\dots\dots\dots \\
\qquad\qquad\qquad\qquad +2^2+2^1+1+1 \\
\qquad\qquad\qquad\qquad\quad +2^1+1+1 \\
\qquad\qquad\qquad\qquad\qquad +1+1+2 \\
=2^{n-r-4}+2^{n-r-3}+2^{n-r-2}+\dots\dots+2^2+2+2 \\
=2^{n-r-3}
\end{array}
$$

Comparing this result with the expression for ϕ_1 we find

$$
\Delta_1=\phi_1\div\left\{\phi_3\,\phi_4^2\,\phi_5^3\dots\dots\phi_{n-1}^{n-3}\,\phi_n^{n-2}\right\}
$$

$$
=\phi_1^1\,\phi_2^0\,\phi_3^{-1}\,\phi_4^{-2}\,\phi_5^{-3}\dots\dots\phi_{n-1}^{-n+3}\,\phi_n^{-n+2}.
$$

Thus the general determinant of nth order is expressed in terms of compound determinants of the 2nd order.

Theorem regarding Orthogonal Conics.

BY WILLIAM FINLAYSON.

(*Read 20th January. MS. received, 14th May, 1908.*)

DEFINITION. When two conics intersect each other at two points in such a manner that the tangents and normals of the one become the normals and tangents of the other, they may be said to cut each other orthogonally.

Theorem. 1st, A given conic can be cut at every point on it by two conics which are orthogonal to it; 2nd, every conic orthogonal to a given conic passes through two fixed points on the axis of the given conic.

1. Let F and S, Figure 4, be the foci of the given conic, Xx and Yy the directrices, and let R equal the radius of director circle, centre F, and let P be any point on the curve. Draw PSP_1 a focal chord; then Sx, at right angles to PSP_1, cuts the X directrix in x, the centre of the orthogonal circle to F which touches PSP_1 at S, and, since PSP_1, FP and FP_1 are all tangents to this circle, it is therefore the in-circle of triangle PFP_1. Now the normals at P and P_1 bisect the exterior angles of PFP_1 at P and P_1 and the bisectors meet on Fx since Fx bisects angle PFP_1. Calling this point H, we observe that it is the centre of an ex-circle to triangle PFP_1 which touches PSP_1 in S_1 and FP in f^1 so that, if Ff^1 be taken as the radius of the director circle and S_1 as a focus, we get P as a point on an ellipse whose foci are F and S_1 and whose tangent and normal at P are the normal and tangent to the given conic at P; for $Pf^1 = PS_1$ and therefore $FP + S_1P = Ff^1 = R_1$. Similarly $S_1P_1 + FP_1 = Ff^1 = R_1$: therefore the given conic is cut orthogonally at P and P_1 by the ellipse whose foci are F and S_1 and whose directrix is HX_1, a line through H at right angles to FS_1.

The second conic is the Hyperbola whose focal chord in the given conic is FP which cuts the conic in Q and P. Determining Q and taking the normals at Q and P, we can see that they will meet at a point H_1 on the bisector of the exterior angle at S which is the line y_2S, the point y_2 being the centre of the circle of the Y system which was used to determine Q, and S being the focus through which the focal chord QP does not pass. Using the

ex-circle H_1 as the orthogonal circle, its points of contact give us a second focus and the length of the radius of the director circle. The focus in this case, being on the chord QP, is therefore F_1 and the radius of director circle is Ss. We can now construct an orthogonal conic which passes through P and Q; thus there are at any point P on a given conic two intersecting conics orthogonal to the given conic.

2. We have seen in the first part that S and S_1 are the points of contact of the in- and an ex-circle of triangle FPP_1; therefore $S_1P = SP_1$ and $S_1P_1 = SP$, and the semi-latus rectum of the given conic, being an harmonic mean to SP and SP_1, is therefore also an harmonic mean to S_1P_1 and S_1P, and the latus rectum of the orthogonal conic is therefore equal to the latus rectum of the given conic.

Now, for any orthogonal conic, one of the original foci must remain a focus, and therefore FS is always a focal chord: then as in first part we observe that, in triangles S_1II_1 and F_1II_1, the original foci are the points of contact of the in- and an ex-circle to the triangles and that therefore $FI_1 = SI$ and $FI = SI_1$; and, the semi latera recta being equal, and each the harmonic mean to FI_1 and FI and at the same time to SI and SI_1, the points I and I_1 are therefore fixed for all conics orthogonal to the given conic.

NOTES.

(1) As proved in the second part it is to be noticed that the latera recta of all orthogonal conics are equal and that to the latus rectum of given conic.

(2) Since the foci are necessarily internal to both the given conic and the orthogonal conic, these two conics must therefore intersect in four points.

The second pair of cuts are determined by OFO_1 at right angles to HF or $O^1SO^1_1$ at right angles to H_1S, since FO, for instance, is the common polar of F to the touching orthogonals at y_2 and H^1 which determine O and O_1 for both curves.

(3) The normals at P and P_1 are parallel to Sf and Sf_1, and the chord FO is parallel to ff_1, being respectively at right angles to the same line.

(4) The second pair of intersections cannot be orthogonal for FOS and FOS_1 cannot have the same bisector.

(5) Conics to be orthogonal must have two foci on a common focal chord, the remaining two coinciding, as for instance in the given conic and the orthogonal ellipse S_1 and S lie on a common focal chord while F is the double focus.

(6) Let R be the radius of director circle R_1 and R_2 the radii of the director circles of the orthogonal conics then the following simple relation exists for them :—

$R_1 = Ff^1 = R + fP + Pf^1 = R + SP + SP_1 = R + PP_1$ for orthogonal ellipse, while from $SQ - FQ = R$ (1) $F_1Q - SQ = R_2$. (2) $FP - SP = R$ (3) and $SP - F_1P = R_2$ (4) by adding (1) and (2), (3) and (4), and then adding, we get $QP = R + R_2$, or $R_2 = QP - R$ in the case of the orthogonal Hyperbola.

(7) When the focal chord common to both is at right angles to FS, then S_1 coincides with S and (6) becomes $R_1 = R + LL_1$, LL_1 being the latus rectum. Halving this becomes $CI = CA + SL$; but $CI = CA + AI$, therefore $AI = SL$, and, if J be the point where the axis of the ellipse cuts the given conic, then by above $JA_1 = AI = SL$. So that if the ellipse be regarded as the given conic, J and J_1 are the fixed points through which all orthogonals pass.

(8) CC_1 is parallel to PP_1; CC_2 is parallel to QP.

The Dygogram of Axle Reaction of a Pendulum

By A. G. GREENHILL

(*Read* 13*th December*, 1907. *Received, same date.*)

Dygogram is the name of a curve, invented by Archibald Smith and employed in his Admiralty *Manual of the Deviation of the Compass*, to give a graphic representation of the varying magnetic field of a compass as the ship is swung round in azimuth; a description is given of the Dygogram by Maxwell in *Electricity and Magnetism*, § 441.

There are two kinds of the Dygogram; one gives the magnetic field with respect to the land as the ship turns in azimuth, the other gives the magnetic field relatively to the ship.

The curves are drawn on two cards, superposed and pivoted at a point; and thereby a reading of the compass on the ship can be converted into a true magnetic bearing with respect to the land or chart.

Archibald Smith's Compass Dygogram of the first kind is a Limaçon of Pascal, and the second kind is an ellipse; we may distinguish them as the space and body dygogram.

Now it is curious that the same curve, the limaçon and ellipse, will serve as the dygogram, space and body, for the axle reaction of a pendulum, swinging freely under gravity about a fixed axis.

The investigation is important in the case of a large body, such as a ship, a ballistic pendulum, a church bell, a hammock chair, or a swing at a fair, for calculation of the strength and stability of the frame; and Kater's experimental determination of g by his pendulum is now considered untrustworthy from insufficient consideration of the yielding of the support employed.

The reaction at the axle is obtained at once by replacing the pendulum by two particles kinetically equivalent, in the manner explained in Maxwell's *Matter and Motion*, CXXI; one particle of weight $W\frac{h}{l}$ is placed at centre of oscillation L, and the other of weight $W\left(1-\frac{h}{l}\right)$ at O the centre of suspension, W denoting the

weight of the pendulum, l the length OL of the equivalent simple pendulum, and h the distance OG from O of G the centre of gravity.

These two particles have the same weight W as the pendulum, the same centre of gravity at G, and the same moment of inertia about G

$$(1) \quad W\frac{h}{l}(l-h)^2 + W\left(1-\frac{h}{l}\right)h^2 = Wh(l-h) = Wk^2,$$

since (2) $$l = h + \frac{k^2}{h},\ l - h = \frac{k^2}{h},$$

k denoting the radius of gyration of the pendulum above an axis through G parallel to the axle through O; and so the two particles, if connected rigidly, will in uniplanar motion behave under any force in the same manner as the pendulum.

Of these two particles, the second one acts by dead weight at O, where it is at rest; the first particle pulls at the axle with a force in OL, composed in the gravitation unit of $W\frac{h}{l}\cos\theta$ due to gravity, and $W\frac{h}{l}\ \frac{l\omega^2}{g} = W\frac{h\omega^2}{g}$ due to centrifugal force of an angular velocity ω; the total pull is then

$$(3) \quad P = Wh\left(\frac{\cos\theta}{l} + \frac{\omega^2}{g}\right).$$

Suppose the pendulum to be swinging through an angle $2a$, so that, by the Principle of Energy,

$$(4) \quad \tfrac{1}{2}l\omega^2 = g(\cos\theta - \cos a)$$

$$(5) \quad P = W\frac{h}{l}(3\cos\theta - 2\cos a),$$

and the resultant of P in OL and the dead weight $W\left(1-\frac{h}{l}\right)$ vertical at O is the pull of the pendulum on the axle.

Representing W to scale by the length l, then P is given to the same scale by the vector r of the curve DP in Figure 5, in which

$$(6) \quad r = h(3\cos\theta - 2\cos a),$$

a limaçon; and if this curve is drawn with the pole H vertically below O at a depth $OH = GL = l - h$, the pull on the frame of support is given by the vector OP; the limaçon DP is thus the

space, or frame, Dygogram of axle reaction, with respect to a pole at O; and as the pendulum swings so that θ changes from 0 to a, the vector r diminishes from $h(3-2\cos a)$ to $h\cos a$.

Draw PQR vertically upward meeting OL in Q, and make $QR=3h$; draw RC perpendicular to OL, then $CQ=3h\cos\theta$, while $OQ=HP=3h\cos\theta-2h\cos a$, so that in Figure 5, where a is an obtuse angle, $OC=-2h\cos a$; and C is the centre of an ellipse APB described by P in the pendulum, in which $PQ=OH=l-h$, and $PR=l+2h$ are the semi-axes, minor and major.

The vector PO then represents the pull of the frame on the pendulum, so that the body Dygogram with respect to a pole at O is an ellipse always of the same size, but with the centre C in OL at a distance $OC=-2h\cos a$, depending on a the amplitude of oscillation.

The normal Pg' of the ellipse passes through g' the intersection of Qg' parallel to CB and Rg' parallel to CA; or else the tangent VP is given by making $CV=K\gamma=(l-h)\operatorname{cosec}\theta$, where $G\gamma$ is parallel to LK.

The normal Pg of the limaçon is obtained by making Hg equal and parallel to CR.

If a slot CQ and CR is cut on the pendulum, and the point Q and R on a rod PQR is guided in them as an elliptic compass, so as to be kept vertical by mechanism, the point P will describe the body dygogram ellipse in the pendulum, and the space dygogram limaçon in the frame of support.

At the end of a swing, where

$$\theta=a,\ P=W\frac{h}{l}\cos a, \tag{7}$$

so that P is negative if $a>90°$, and the thread of a simple pendulum OL would become slack; $a=90°$ makes the frame dygogram a circle, and the body dygogram ellipse has a centre at O.

When the pendulum just reaches the highest position, $a=180°$, $\cos a=-1$; and after the pendulum passes the highest position and makes a complete revolution, $-\cos a$ is to be replaced by $\sec a$, so that

$$P=W\frac{h}{l}(3\cos\theta+2\sec a); \tag{8}$$

and when $2\sec\alpha = 3$, $\alpha = 48° 12'$, the reaction P is zero at the highest position and $6W\frac{h}{l}$ in the lowest position, and the frame dygogram is a cardioid.

For a larger value of α the frame dygogram is a non-nodal limaçon, so that P does not change sign, as required in a centrifugal railway, where the apparent gravity is nearly zero in the highest position, and so in the lowest position is more than sixfold the ordinary g; the carriage does not then leave the track, or the passengers feel any tendency to falling out.

The same conditions are required when water in a tumbler is whirled round in a vertical circle.

If the vector OP of the body dygogram of a swinging placard or picture falls outside the angle formed at the supporting nail by the two cords, it implies that one of the cords has become slack. So also the vector OP of the frame dygogram of a swing should lie for safety inside the angle formed by the A frame of the supports of the axle.

Measure OI vertically upward of length $\lambda = \frac{g}{\omega^2}$, so that λ is the height of the equivalent conical pendulum, which swings round the vertical with angular velocity ω; then from (4)

$$(9) \qquad \lambda = \frac{\frac{1}{2}l}{\cos\theta - \cos\alpha},$$

so that for a swinging pendulum the body locus of I is a hyperbola, with focus at O and direction lk cutting LO produced at right angles in l, where $Ol = \frac{1}{2}$LO; the eccentricity of the hyperbola is $\sec\alpha$, and tI is the tangent at I, if KO, kl intersect in t.

Describe a circle, centre I and radius IO, cutting lk in k; then Ik is parallel to an asymptote of the hyperbola, and so makes an angle α with OL.

This hyperbola becomes the straight line lk when $\alpha = 90°$; and a parabola when $\alpha = 180°$, and the pendulum just reaches the highest position; afterwards, with complete revolutions, the curve changes into an ellipse.

Draw OK horizontal and LK at right angles to OL to meet in K; draw OJ at right angles to IK and denote the angle IOJ or OKI by ϕ.

Then if $\dot{\omega}$ denotes the angular acceleration of the pendulum, and ω as before the angular volocity,

$$(10) \qquad \dot{\omega} = \frac{g}{l}\sin\theta = \frac{g}{\mathrm{OK}},\ \omega^2 = \frac{g}{\mathrm{OI}},$$

$$(11) \qquad \frac{\dot{\omega}}{\omega^2} = \frac{\mathrm{OI}}{\mathrm{OK}} = \tan\phi\,;$$

and the reversed acceleration, or kinetic reaction, at a point P anywhere on the pendulum is the resultant of $\dot{\omega}$ OP perpendicular to OP and ω^2.OP along PO, and is therefore

$$(12) \qquad \omega^2.\mathrm{OP}\sec\phi = g\frac{\mathrm{OP}}{\mathrm{OI}\cos\phi} = g\frac{\mathrm{OP}}{\mathrm{OJ}},$$

in a direction making an angle ϕ with OP; and the field of kinetic reaction is thus composed of equiangular spirals of angle ϕ, with pole common at O. Thus the reversal acceleration of J is g, vertically upward.

Combining this field with the uniform field of gravity g vertical, or making an angle ϕ with OJ, the resultant field is of strength $g\dfrac{\mathrm{JP}}{\mathrm{OJ}}$ making an angle ϕ with JP, so that the compound field of gravity and kinetic reaction is composed of equiangular spiral lines of force of radial angle ϕ, round the common pole J.

This is the field of force experienced by a body fixed at any point P of the pendulum; it can be shown experimentally by the varying level of coloured liquid in a small flask fixed at P, or by a plummet at the end of a short plumb line thread.

Produce OJ to meet in J′ the vertical GNJ′ through G, cutting OK in N; then

$$(13) \qquad \mathrm{OJ}'.\mathrm{OJ} = \mathrm{OK}.\mathrm{ON} = \mathrm{OL}.\mathrm{OG} = lh = h^2 + k^2 = k_1^2,$$

where k_1 denotes the radius of gyration about the axle; thus J and J′ describe inverse curves with respect to the pole O, either in the pendulum or frame.

Drawing J′N′ at right angles to the vertical through O,

$$(14) \qquad \mathrm{ON}'.\mathrm{OI} = \mathrm{OJ}'.\mathrm{OJ} = lh,$$

$$(15) \qquad \mathrm{ON}' = \frac{lh\,(\cos\theta - \cos\alpha)}{\frac{1}{2}l} = 2h\,(\cos\theta - \cos\alpha),$$

and making OC′ = OC,

$$(16) \qquad \mathrm{C'N'} = 2h\cos\theta,\ \mathrm{N'J'} = h\sin\theta,$$

so that the frame locus of J′ is the ellipse, centre C′ and semi-axes $2h$ and h ; and since

$$(17) \qquad GJ' = GN + ON' = h\,(3\cos\theta - 2\cos\alpha) = HP,$$

the body locus of J′ is a limaçon, equal to the frame dygogram DP.

The sensation of the varying compound field is experienced most forcibly on a rolling ship, as shown in Figure 6 by a midship cross section of the Caronia, from keel to flying bridge; the ship is supposed to be rolling through 90°, to $\alpha = 45°$ on each side of the vertical; it is drawn when heeled at an angle $\theta = 22°\frac{1}{2}$.

Considered as a pendulum, the ship may be treated as if supported at G, and acted on by the buoyancy, Wtons, applied vertically upward at the metacentre M, at a constant metacentric height GM.

The direction of the plumb line is drawn at various positions of P, showing the apparent field of force; if on the flying bridge it will be seen that a body would be left behind in the air if not lashed down, and water would not stay in a tank if open.

Proceeding downward in the ship the motion becomes easier; at G the field of force is undisturbed g vertical; at L the centre of oscillation the direction of the field is along GL, and no motion is felt, or straphanging required, so that water in a vessel at L will not wash about, and the water in a boiler will be almost steady if the surface is not far from L.

A swing is of little use as a preparation for a voyage, with the seat in the neighbourhood of L; it must be made like the cross section in Figure 6, with access to the upper parts as high as possible above the axle of suspension.

A model is in use at Woolwich made of canvas stretched on a frame work of battens to resemble figure 2; suspended against the wall by a pivot at M, the frame can roll like a pendulum, and short plumb lines or small vessels of coloured liquid attached at various points will show the disturbance of the field of apparent gravity.

The canvas is painted to resemble the interior of the ship, like the coloured advertisements; the cross section of the Kaiser Wilhelm II. was selected, and made to one-tenth scale; the frame is about 8 feet wide and 12 feet high.

If our Figure 6 represents a ship of 80 feet beam and 35 draft, then GL represents 30 feet, and the time of a single roll from side to side would be about 3 seconds; such a motion would be much too violent for the ship to be habitable, although not unsafe.

The metacentric height GM, about 10 feet as shown in Figure 6, would require to be reduced to about one foot, making GL 300 feet, and the time of a single roll about 9·5 seconds; and now the motion would be comfortable for moderate angle of roll.

A question bearing on the subject proposed by Sir W. Thomson is of historical interest, given in the Mathematical Tripos of 1874, on Thursday afternoon, January 8, and so it is transcribed here at full length. "In the motion of a compound pendulum prove that

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cos\theta - \cos a),$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta,$$

showing what l is in terms of properties of the solid and the position of the axis.

Find the force which acts on any infinitesimal part of the whole mass, to balance its weight and give it its acceleration. Hence show that if an infinitely small mass be hung by an infinitely short cord from any point (x, y) of the pendulum, the indication of the cord to x will be

$$\tan^{-1}\frac{-g\sin\theta - \dot{\omega}\,x + \omega^2 y}{g\cos\theta + \dot{\omega}\,y + \omega^2 x},$$

the axis of x being the line through the point of support and the centre of gravity of the pendulum; and ω, $\dot{\omega}$ being used to denote $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ respectively.

Supposing a ship to roll round a horizontal axis according to the same law as that of the vibration of a compound pendulum, find the position of the axis and the angle (great or small) through which the ship rolls; from observations on two plummets hung by very short threads at a given distance from one another

in a line through the axis perpendicular to the deck, and a determination of the period of rolling in cases where the angle is small."

The Dygogram can be illustrated experimentally by balancing an object on the floor or table, like a poker, a ruler, a circular protractor, or a pin; letting it fall over, and observing whether the contact is broken.

Changing the measurement of a and θ to deviation from the upward vertical, the height of the shoulder of the frame Dygogram limaçon above H is $\frac{1}{3}h\cos^2 a$, and then $r = h\cos a$, $\cos\theta = \frac{1}{3}\cos a$; and O lies below the shoulder where

$$(18) \qquad l - h = \frac{k^2}{h} = r\cos\theta = h\,(2\cos a\cos\theta - 3\cos^2\theta)$$

$$(19) \qquad 3\cos\theta = \cos a \pm \sqrt{\left(\cos^2 a - 3\frac{k^2}{h^2}\right)}$$

$$(20) \qquad r = h\cos a \mp \sqrt{(h^2\cos^2 a - 3k^2)}.$$

This does not occur unless

$$(21) \qquad h^2\cos^2 a > 3k^2,$$

and between these values of r and θ the vector OP of the Dygogram points upward, and contact is broken unless the axle is held down by the cap of a bearing, as in a church bell.

For a uniform rod, like a flat ruler, $k^2 = \frac{1}{3}h^2$, so that O is at the level of the shoulder of the limaçon when $a = 0$, and the ruler is balanced on one end in the vertical position, and then the contact is broken for an instant when $\cos\theta = \frac{1}{3}$, $\theta = 70°\frac{1}{2}$.

A circular disc, like a protractor, $k^2 = \frac{1}{4}a^2$, just breaks contact if let fall from an angle $a = 30°$; if balanced vertically, contact is broken when θ reaches 60°.

A pin will show a difference according as it is balanced on the point or the head, and then let fall. The demolition of a tall chimney stalk will provide an illustration on a large scale, as to whether contact with the ground is broken, or not.

With $h = l$, the pendulum becomes the simple pendulum of a thread OL and plummet L, and O then coincides with H, the node of the Dygogram limaçon.

The thread becomes slack when $\cos\theta = \frac{2}{3}\cos a$; but this does not happen unless $a > 90°$, or for complete revolutions when $\cos\theta = -\frac{2}{3}\sec a$.

The plummet then proceeds to describe a parabola, osculating the circle; and the thread becomes tight again where the circle and parabola intersect, which will be on the isoclinal chord; so that if θ is the angle the thread makes with the upward vertical when the thread becomes slack, 3θ is the angle when the thread tightens again.

The extension of the Dygogram is not difficult to the case of a symmetrical top, spinning with the point in a smooth cup, or suspended from a universal joint; an additional component force is introduced of magnitude

$$\text{(22)} \qquad W\frac{C}{A}\,\frac{hR\omega}{g},$$

along the vector of ω the component angular velocity perpendicular to the axis of figure of the top, where C, A denote the moment of inertia about the axis of the top and about a perpendicular axis through the point, and R denotes the angular velocity of the spin.

A similar formula holds if the pendulum contains a flywheel, and it will serve to explain the theory of the Schlick apparatus for mitigating the rolling of a ship.

A Graphic Method of Solving n Simultaneous Linear Equations involving n Unknowns

BY R. F. MUIRHEAD

(*Read 13th March, 1908. Received, same date.*)

§ 1.

The method here explained affords a complete solution of the problem to determine by geometric construction the values of any number of unknowns connected by an equal number of equations of the first degree. The construction consists entirely of straight lines, and can be carried out by the aid of a straight-edge and a scale; or, in its modified form, in which parallel lines are required, by these instruments along with a set-square.

§ 2.

THE CASE OF TWO SIMULTANEOUS EQUATIONS

$$a_1 x + b_1 y + c_1 = 0 \qquad (1)$$
$$a_2 x + b_2 y + c_2 = 0 \qquad (2)$$

Draw a triangle OXY, and in OX take X_1, X_2, so that

$$a_1 \mathrm{OX}_1 + c_1 \mathrm{X}_1\mathrm{X} = 0$$
$$a_2 \mathrm{OX}_2 + c_2 \mathrm{X}_2\mathrm{X} = 0$$

Similarly, in OY take the points Y_1, Y_2 so that

$$b_1 \mathrm{OY}_1 + c_1 \mathrm{Y}_1\mathrm{Y} = 0$$
$$b_2 \mathrm{OY}_2 + c_2 \mathrm{Y}_2\mathrm{Y} = 0$$

Let $X_1 Y_1$ meet $X_2 Y_2$ in P
" YP " OX " X_{12}
" XP " OY " Y_{12}

Then $x = \mathrm{OX}_{12}/\mathrm{X}_{12}\mathrm{X}$; $y = \mathrm{OY}_{12}/\mathrm{Y}_{12}\mathrm{Y}$
are the values which satisfy (1) and (2)

To verify that they satisfy (1) we may proceed thus:—

$$a_1 \frac{\mathrm{OX}_{12}}{\mathrm{X}_{12}\mathrm{X}} + b_1 \frac{\mathrm{OY}_{12}}{\mathrm{Y}_{12}\mathrm{Y}} + c_1 = c_1\left(- \frac{\mathrm{X}_1\mathrm{X}}{\mathrm{OX}_1} \cdot \frac{\mathrm{OX}_{12}}{\mathrm{X}_{12}\mathrm{X}} - \frac{\mathrm{Y}_1\mathrm{Y}}{\mathrm{OY}_1} \cdot \frac{\mathrm{OY}_{12}}{\mathrm{Y}_{12}\mathrm{Y}} + 1\right)$$
$$= c_1\left(- \frac{\mathrm{XP}}{\mathrm{PY}_{12}} \cdot \frac{\mathrm{Y}_{12}\mathrm{Y}_1}{\mathrm{Y}_1\mathrm{O}} \cdot \frac{\mathrm{PY}_{12}}{\mathrm{XP}} \cdot \frac{\mathrm{YO}}{\mathrm{Y}_{12}\mathrm{Y}} - \frac{\mathrm{Y}_1\mathrm{Y}}{\mathrm{OY}_1} \cdot \frac{\mathrm{OY}_{12}}{\mathrm{Y}_{12}\mathrm{Y}} + 1\right)$$

(applying Menelaus' Theorem to the transversals $X_1 PY_1$ and $X_{12} PY$ of the triangle OXY_{12})

$$=c_1\left(-\frac{Y_{12}Y_1}{Y_1O}\cdot\frac{YO}{Y_{12}Y}-\frac{Y_1Y}{OY_1}\cdot\frac{OY_{12}}{Y_{12}Y}+1\right)$$

$$=c_1\frac{Y_{12}Y_1 \, . \, YO+YY \, . \, OY_{12}+OY_1 \, . \, Y_{12}Y}{OY_1 \, . \, Y_{12}Y}=0.$$

Similarly we can show that the same values of x and y satisfy (2).

The process of constructing the points X_{12}, Y_{12} when the points X_1, Y_1, X_2, Y_2 are given on the triangle OXY may be called for brevity "solving $X_1 Y_1$ and $X_2 Y_2$ graphically, as to x and y", the points X_{12}, Y_{12} being called "the solution".

§ 3.

THE CASE OF THREE EQUATIONS

$$a_1 x+b_1 y+c_1 z+d_1=0 \qquad - \quad - \quad - \quad - \qquad (3)$$
$$a_2 x+b_2 y+c_2 z+d_2=0 \qquad - \quad - \quad - \quad - \qquad (4)$$
$$a_3 x+b_3 y+c_3 z+d_3=0 \qquad - \quad - \quad - \quad - \qquad (5)$$

Draw any tetrastigm OXYZ, and let X_1, Y_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 be determined in the same way as X_1, Y_1, X_2, Y_2, were before; *e.g.* X_1 in OX makes $a_1 OX_1/X_1 X+d_1=0$ and Z_2 in OZ makes $c_2 OZ_2/Z_2Z+d_2=0$

Then solve

$X_1 Z_1$, $X_2 Z_2$	graphically	as to x, z,	and	let	X_{12}, Z_{12}	be the	solution
$X_1 Z_1$, $X_3 Z_3$	"	"	"	"	X_{13}, Z_{13}	"	"
$Y_1 Z_1$, $Y_2 Z_2$	"	" y, z	"	"	Y_{12}, Z_{12}	"	"
$Y_1 Z_1$, $Y_3 Z_3$	"	"	"	"	Y_{13}, Z_{13}	"	"

Next solve X_{12} Y_{12}, X_{13} Y_{13} graphically as to x, y and let X_{123} Y_{123} be the solution. In a similar manner, by interchanging, say X and Z, we can determine Y_{123}, Z_{123}.

Then $x=OX_{123}/X_{123}X$, $y=OY_{123}/Y_{123}Y$, $z=OZ_{123}/Z_{123}Z$ are the values of x, y, z satisfying the equations (3), (4), (5).

§ 4.

THE CASE OF n EQUATIONS

The extension of the method to any number of equations will now be obvious.

If, for example, we have worked the method up to the solution of $n-1$ simultaneous equations, then for the case of n equations we first solve graphically as to x and y two sets of $n-1$ equations each, viz.

(i) Equations (1), (3), (4)......(n) giving solutions $X_{134\ldots n}$, $Y_{134\ldots n}$
(ii) „ (1), (2), (4)......(n) „ „ $X_{124\ldots n}$, $Y_{124\ldots n}$

Then solve graphically as to x and y the four points thus determined on OX and OY, giving the solution $X_{123\ldots n}$, $Y_{123\ldots n}$. Then the values $x = OX_{123\ldots n}/X_{123\ldots n}X$, $y = OY_{123\ldots n}/Y_{123\ldots n}Y$, form part of the solution of the n given equations; and the values of the other unknowns can be worked out in a similar manner.

It is clear that a great variety of constructions exist which can be got by interchanging pairs of letters or of suffixes, and that a limited number of these is sufficient to determine all the unknowns.

§ 5.

Let us now consider the *rationale* of the method. It is obviously suggested by the well-known simple method of solving two simultaneous equations of the first degree by the intersection of two linear graphs. The rule for three simultaneous equations was arrived at by considering the projection upon a plane of a figure consisting of a tetrahedral coordinate system and its intersections with three planes representing the three given equations. But it is exactly parallel to the algebraic method of eliminating one unknown in two ways so as to get two equations involving only two unknowns which are then solved by eliminating successively the two remaining unknowns. And this parallel holds in the general case. It will be sufficient to point out the parallel in detail in the case of four equations with four unknowns.

We have $a_r\dfrac{OX_r}{X_rX}+e_r=0$, &c. (where $r=1, 2, 3, 4$) so that the rth equation can be put in the form

$$\frac{x}{OX_r/X_rX}+\frac{y}{OY_r/Y_rY}+\frac{z}{OZ_r/Z_rZ}+\frac{u}{OU_r/U_rU}=\rho$$

Where ρ is the letter introduced to make the equations homogeneous, to which the value 1 is given, then the points X_{12}, Y_{12}, Z_{12}, &c., are such that

$$\frac{x}{OX_{12}/X_{12}X} + \frac{y}{OY_{12}/Y_{12}Y} + \frac{z}{OZ_{12}/Z_{12}Z} = \rho$$

$$\frac{x}{OX_{13}/X_{13}X} + \frac{y}{OY_{13}/Y_{13}Y} + \frac{z}{OZ_{13}/Z_{13}Z} = \rho$$

$$\frac{x}{OX_{14}/X_{14}X} + \frac{y}{OY_{14}/Y_{14}Y} + \frac{z}{OZ_{14}/Z_{14}Z} = \rho$$

and further

$$\frac{x}{OX_{123}/X_{123}X} + \frac{y}{OY_{123}/Y_{123}Y} = \rho, \quad \frac{x}{OX_{124}/X_{124}X} + \frac{y}{OY_{124}/Y_{124}Y} = \rho$$

and lastly $$\frac{x}{OX_{1234}/X_{1234}X} = \rho, \quad \frac{y}{OY_{1234}/Y_{1234}Y} = \rho$$

§ 6.

Modified Construction.

By sending the points XYZ ... off to infinity, while O remains near, we get an extreme case of the general method. The only modifications entailed in the construction are first, that OX_1 is cut off from OX to be equal to $-\frac{c_1}{a_1}$, instead of OX being divided in that ratio, and second, that when such a pair of lines as X_1Y_1, X_2Y_2 intersect in P, X_{12} is got by drawing through P a line parallel to OY, instead of passing through Y.

We might further modify the construction by using only two axes of reference, say OA, OB, and letting each step of the graphic solution be worked on these axes. This would involve the transference of some of the points from one axis to another. For instance, in § 2, if OX_1, OX_2, OX_3 were cut off from OA and OZ_1, OZ_2, OZ_3 from OB then the points X_{12}, X_{13} would fall on OA; if then the same process were carried through with Y instead of X, we should have the points Y_{12}, Y_{13} also on OA, and they would have to be transferred to OB before we could graphically solve X_{12}, Y_{12}, X_{13}, Y_{13}.

§ 7.

Let us now estimate the least number of lines required in addition to OX, OY, OZ, &c., to find *one* unknown, say x, from n simultaneous linear equations.

For $n = 2$ the number is clearly $2 + 1 = 3$

For $n = 3$, if we proceed as in § 3, each of the three points Z_1, Z_2, Z_3 must be joined to 2 points (6 lines) giving 4 intersections through which other 4 lines have to be drawn; and the final stage requires 3 more lines; so that the total for $n = 3$ is 13 lines.

For $n = 4$, the number of lines required in the first stage will be $4 \times 3 + 9$, in the second stage 13, and in the third 3. Total 34.

A little consideration will show that in the general case the number of lines required to find x is $3 + 2 . 5 + 3 . 7 + \ldots\ldots + (n-1)(2n-1)$, which amounts to $n(n-1)(4n+1)/6$, or if we include the n lines of reference OX, OY, &c., the total is $n(4n^2 - 3n + 5)/6$.

§ 8.

Verification of the Method by the aid of Determinants.

Let us take for simplicity the modified method in which X, Y, Z...are at infinity and (for § 1) $OX_1 = -\dfrac{c_1}{a_1}$ &c.

In § 1 the solution of (1) and (2) is given by

$$x = \begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = (b_1 c_2) \div (a_1 b_2)$$

$$y = -\begin{vmatrix} a_1 c_1 \\ a_2 c_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = -(a_1 c_2) \div (a_1 b_2)$$

Hence, in § 2, since

$$\begin{aligned} OX_{12} &= (c_1 d_2) \div (a_1 c_2) \\ OX_{13} &= (c_1 d_3) \div (a_1 c_3) \\ OY_{12} &= (c_1 d_2) \div (b_1 c_2) \\ OY_{13} &= (c_1 d_3) \div (b_1 c_3) \end{aligned}$$

the equation to $X_{12}\,Y_{12}$ is $\quad \dfrac{x(a_1 c_2)}{(c_1 d_2)} + \dfrac{y(b_1 c_2)}{(c_1 d_2)} = 1$

and that to $X_{13}\,Y_{13}$ is $\quad x(a_1 c_3) + y(b_1 c_3) = (c_1 d_3)$

Hence $\quad OX_{123} = \begin{vmatrix} (c_1 d_2), & (b_1 c_2) \\ (c_1 d_3), & (b_1 c_3) \end{vmatrix} \div \begin{vmatrix} (a_1 c_2), & (b_1 c_2) \\ (a_1 c_3), & (b_1 c_3) \end{vmatrix}$

Now, if we denote by $A_1\ A_2 \ldots$ the co-factors of $a_1, a_2 \ldots$ in the determinant $(a_1\, b_2\, c_3)$ the determinant after the $\div$ sign is $= -\begin{vmatrix} B_3 & A_3 \\ B_2 & A_2 \end{vmatrix} = -\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}$ and therefore, by the theory of reciprocal determinants and their minors, is $= -\ c_1\,(a_1\, b_2\, c_3)$. Similarly, the determinant before the sign $\div\ = +c_1\,(a_1\, b_2\, c_3) = +c_1\,(b_1\, c_2\, d_3)$

Thus
$$\begin{aligned} OX_{123} &= -\ (b_1\, c_2\, d_3) \div (a_1\, b_2\, c_3) \\ &= -\ (d_1\, b_2\, c_3) \div (a_1\, b_2\, c_3) \end{aligned}$$

which we know to be the value of x got from equations (3), (4), (5).

To extend the proof, by mathematical induction, to the general case of n equations, we assume

$OX_{123\ldots(n-1)} = -(h_1\, b_2\, c_3 \ldots g_{n-1}) \div (a_1\, b_2\, c_3 \ldots g_{n-1})$, and then, starting with n equations we have

$OX_{123\ldots(n-1)} = -(h_1\, c_2\, d_3 \ldots h_{n-1}) \div (a_1\, c_2\, d_3 \ldots h_{n-1})$

$OX_{123\ldots(n-2)n} = -(k_1\, c_2\, d_3 \ldots g_{n-2} h_n) \div (a_1\, c_2\, d_3 \ldots g_{n-2} h_n)$ and two similar results with X and Y, a and b interchanged.

Then
$$OX_{12\ldots\ldots n} = \begin{vmatrix} -(k_1\, c_2\, d_3 \ldots h_{n-1}), & (b_1\, c_2\, d_3 \ldots h_{n-1}) \\ -(k_1\, c_2\, d_3 \ldots g_{n-2} h_n), & (b_1\, c_2\, d_3 \ldots g_{n-2} h_n) \end{vmatrix}$$
$$\div \begin{vmatrix} (a_1\, c_2\, d_3 \ldots h_{n-1}), & (b_1\, c_2\, d_3 \ldots h_{n-1}) \\ (a_1\, c_2\, d_3 \ldots g_{n-2} h_n), & (b_1\, c_2\, d_3 \ldots g_{n-2} h_n) \end{vmatrix}$$

The numerator
$$= \begin{vmatrix} B_n(-1)^n, & A_n(-1)^{n-1} \\ B_{n-1}(-1)^{n-1}, & A_{n-1}(-1)^n \end{vmatrix} = \begin{vmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{vmatrix}$$

when A_n is the co-factor of a_n in the determinant $(a_1\, b_2 \ldots h_n)$, &c. By the theory of Reciprocal Determinants this gives us $(c_1\, d_2 \ldots h_{n-2})\ (a_1\, b_2\, c_3 \ldots h_n)$ similarly the denominator reduces to $-(c_1\, d_2 \ldots h_{n-2})\ (k_1\, b_2\, c_3 \ldots h_n)$

Hence $OX_{12\ldots\ldots n} = -\ (k_1\, b_2\, c_3 \ldots\ldots h_n) \div (a_1\, b_2 \ldots\ldots h_n)$, which completes the proof.

§ 9.

To test the practicability of the graphic method explained in this example, I worked out on an ordinary piece of squared paper, 7 inches by 9, with very ordinary instruments, the graphic solution of the following set of four simultaneous equations.

$$\begin{aligned} x + \tfrac{y}{3} + \tfrac{z}{2} + \tfrac{u}{2} &= 1 \\ \tfrac{x}{2} + y + z - u &= 1 \\ 2x - 2y - 2z - \tfrac{u}{2} &= 1 \\ -\ x + \tfrac{y}{2} - \tfrac{z}{3} + \tfrac{u}{4} &= 1 \end{aligned}$$

The numerical process of solution gives $x = \frac{584}{529} \doteqdot 1{\cdot}104$, $y = \frac{1437}{529} \doteqdot 2{\cdot}72, z = -\frac{1134}{529} \doteqdot -2{\cdot}15, u = \frac{66}{529} \doteqdot {\cdot}125$. The graphic process gave $x = 1{\cdot}14$, $y = 2{\cdot}8$, and again $x = 1{\cdot}2$, $z = -1{\cdot}95$. The axes were taken so as to divide space into octants, each of the lines OX, OY, OZ, OU making an angle of 45° with the one preceding it. The second value of x, got by solving for x and z, is less accurate than the first value, and this is, probably, due to the fact that the points X_{124} Z_{124} come very close together, giving an ill-conditioned determination of the line joining them. Some of the intersections lay several inches beyond the paper, and additional pieces of paper were temporarily used. This might have been avoided by using one of the well-known methods of drawing a line through a given point or in a given direction, towards a point of intersection of two lines when it lies out of reach. I did not time myself, either during the construction or during the calculation, but my impression is that the latter took more time—and certainly it would do so in the case of equations with less simple numerical coefficients. Against this possible gain there must be set the very limited degree of accuracy of the graphical method. But I believe this method may have useful applications to a good many technical problems.

[*Added 18th June, 1908.*] After the foregoing paper was in print, my attention was called by the Editor to a Note on the same topic by Mr F. Boulad in Vol. 7 of the 4th Series of the "Nouvelles Annales," and I find by reference to the "Encyclopædie der Math." that several papers on the subject have appeared, the most important of which seems to be that of Van den Berg (Verslagen en Mededeelingen der koninklijke Akademie van Wetenschappen, Amsterdam, 1888.) The only article on the subject I have found published in English is that by Mr F. J. Vaes in "Engineering," Vol. 66, p. 867 (1898) which explains a method differing considerably from that given above.

Examples in the Geometry of Cross Ratios

By CHARLES TWEEDIE

(*Read 8th May. MS. received 2nd June, 1908*)

When P is joined to four points A, B, C, D coplanar with P, a pencil of four lines is formed whose cross ratio is constant if ABCD are collinear. If A, B, C, D are not in a line the cross ratio P(ABCD) has a value which in general varies with the position of P, but which should be known when P is given in position and also A, B, C, D. A simple expression for the cross ratio is given and its utility in locus problems is illustrated by a variety of simple examples, which in several cases furnish methods for constructing a general cubic curve, with or without double point, a trinodal quartic, etc.

§ 1. Let 1, 2, 3, 4 be four points in a plane, P any fifth point. There exists the following relation in the signed areas of the triangles P12, etc.

$$(P12)\ (P34) + (P13)\ (P42) + (P14)\ (P23) = 0 \qquad \text{I};$$

and the cross ratio of the pencil P(1234) is given by

$$P(1234) = (P13)\ (P24) / (P14)\ (P23) \qquad \text{II}.$$

These are easily established. Let $| P1 | = r_1$, $| P2 | = r_2$, etc.; and let (12) denote the signed angle 1P2.

Then $P12 = \frac{1}{2} r_1 r_2 \sin(12)$; $P34 = \frac{1}{2} r_3 r_4 \sin(34)$; etc.

Substitution in I leads to the trigonometrical identity $\sin(12)\ \sin(34) + \sin(13)\ \sin(24) + \sin(14)\ \sin(23) = 0$ and the right side of II reduces to the well-known expression for the cross ratio $\sin(13)\ \sin(24)/\sin(14)\ \sin(23)$.

Cor. 1. $P(1234) = P(1235) \,.\, P(1254)$;

$\qquad = (P1235)\ P(1256)\ P(1264)$; etc.

Cor. 2. $P(1234) \,.\, P(1342)\ P(1423) = -1$.

Cor. 3. $A(BCPQ) \,.\, B(CAPQ)\ C(ABPQ) = +1$.

Cor. 4. Divide in I by (P14) (P23) when it gives rise to $P(1234) + P(1324) = 1$.

§ 2. Let $P(1234)=\lambda$ it is then easy to establish the following well-known relations either by direct substitution in II or by the aid of *Cor.* 1. and *Cor.* 4. of § (1):

$$P(1234)=P(2143)=P(3412)=P(4321)=\lambda \quad (1)$$
$$P(1243)=1/\lambda, \quad (2)$$
$$P(1324)=1-\lambda, \quad (3)$$
$$P(1342)=1/(1-\lambda), \quad (4)$$
$$P(1423)=1-1/\lambda=(\lambda-1)/\lambda, \quad (5)$$
$$P(1432)=\lambda/(\lambda-1). \quad (6).$$

§ 3. In the dual problem let 1234 be a transversal to the sides of a quadrilateral ABCD as in Figure 7.

Let a, b, c, d be the perpendicular distances of the vertices of ABCD from the transversal.

Denote by $\overline{13}$ the directed segment from 1 to 3, and by (13) the angle between the sides of the quadrilateral that pass through 1 and 3. Let sin(1234) denote sin(13) sin(24)/sin(14) sin(23).

Then
$$A1 \,.\, A3 \sin(13)=a \,.\, \overline{13}=2\triangle A13$$
$$C2 \,.\, C4 \sin(24)=c \,.\, \overline{24}=2\triangle C24$$
$$B1 \,.\, B4 \sin(14)=b \,.\, \overline{14}=2\triangle B14$$
$$D2 \,.\, D3 \sin(23)=d \,.\, \overline{23}=2\triangle D23.$$

Hence
$$\frac{A1}{B1}\cdot\frac{C2}{D2}\cdot\frac{A3}{D3}\cdot\frac{C4}{B4}\times\sin(1234)$$
$$=\frac{ac}{bd}(1234)$$

i.e.
$$\frac{ac}{bd}\sin(1234)=(1234)\quad\left(\because \frac{A1}{B1}=\frac{a}{b};\ \text{etc}\right)$$

Also
$$(A13)(C24)/(B14)(D23)=\frac{ac}{bd}(1234)$$
$$\therefore \frac{A13\,.\,C24}{B14\,.\,D23}\sin(1234)=(1234)^2$$

Cor. From these equations a great variety of identities may be deduced. In particular if $(1234)=-1$ we obtain

$$\frac{(1A3)\ (2C4)}{(1B4)\ (2D3)} \times \frac{\sin(1A3)\ \sin(2C4)}{\sin(1B4)\ \sin(2D3)} = 1.$$

Similarly when ABCD are united into a single point we obtain the formula in areas given in § (1).

§ 4. If P, 1, 2, 3, 4 have co-ordinates (xyz); $(x_1y_1z_1)$; etc., then

$$P(1234) = \frac{(x\,y_1\,z_3)\ (x\,y_2\,z_4)}{(x\,y_1\,z_4)\ (x\,y_2\,z_3)}$$

where $(x\,y_1\,z_3)$ is the determinant whose rows are $x\,y\,z$; $x_1\,y_1\,z_1$; $x_3\,y_3\,z_3$.

For non-homogeneous co-ordinates put $z = 1$ throughout.

In the dual theorem if 1234 is the line $lx + my + n = 0$ and the lines through 1, 2, 3, 4 forming the sides of ABCD are $l_1\,x + m_1\,y + n_1\,z = 0$, etc., then

$$(1234) = \frac{(l\,m_1\,n_3)\ (l\,m_2\,n_4)}{(l\,m_1\,n_4)\ (l\,m_2\,n_3)}.$$

For non-homogeneous co-ordinates put $n = 1$ throughout. The identity of form in these two analytical expressions enables us to dispense with the distinction of co-ordinates.

§ 5. If five points 1, 2, 3, 4, 5 are taken and P be any other point in their plane, there are but two independent cross ratios formed by joining P to four of these points.

Let	$P(1234) = \lambda$; - - - - - - -	I
	$P(1235) = \mu$ - - - - - - -	II
Then	$P(1254) = P(1234)\,/\,P(1235) = \lambda/\mu$ - -	III
	$P(1345) = P(1325)\,/\,P(1324) = (1-\mu)\,(/\,1-\lambda)$	IV
	$P(2345) = P(2315)\,/\,P(2314)$	
	$= \lambda(\mu - 1)\,/\,\mu(\lambda - 1)$ - - -	V

It therefore follows that if *abcd* are any four of the points 12345 then P(*abcd*) is a rational function of λ and μ linear in either of the variables. The elimination of λ and μ from these relations gives rise to a great variety of identical relations.

Cor. The equations I–V also furnish the values of the cross ratio of any four of the five points 1, 2, 3, 4, 5 for the conic containing all the five points.

These may also be verified for the conic as follows. We have the identity

$$5(1234)\,.\,3(1245)\,.\,4(1253)=1$$

If $\lambda=5(1234)$

$\mu=4(1235)$ as on the conic

then $3(1245)\times\lambda\times 1/\mu=1$

Hence $3(1245)=\mu/\lambda$

or $3(1254)=\lambda/\mu$ as in III.

§ 6. If $P(1234)=\lambda$; $P(1235)=\mu$, then P is thereby uniquely determined. For $P(1234)=\lambda$ represents a conic through 1, 2, 3, 4; and $P(1235)=\mu$ a second conic through 1, 2, 3, 5. The conics cut in 1, 2, 3 and in the unique point P. We may therefore speak of λ and μ as being the co-ordinates of P. Only when $P(1234)=5(1234)$; and $P(1235)=4(1235)$ is P indeterminate, being then any point on the conic through 1, 2, 3, 4, 5.

The expression of λ and μ in trilinear co-ordinates may be conveniently found by taking 1, 2, 3 as triangle of reference. Let 4 be the point $(\alpha\,\beta\,\gamma)$, P the point $(x\,y\,z)$.

By the data

$$\lambda=\Delta P13\cdot\Delta P24/\Delta P14\cdot\Delta P23\,;$$

and on making the calculation we find

$$\lambda=(\gamma/z-\alpha/x)/(\gamma/z-\beta/y) \quad . \quad . \quad . \quad (1)$$

Similarly if 5 is the point $(\alpha',\ \beta',\ \gamma')$

$$\mu=(\gamma'/z-\alpha'/x)/(\gamma'/z-\beta'/y) \quad . \quad . \quad (2)$$

Equations (1) and (2) furnish the relations

$$\frac{1}{x}:\frac{1}{y}:\frac{1}{z}=\lambda\,\mu\,(\beta'\gamma-\beta\gamma')-\beta'\gamma\,\mu+\beta\gamma'\lambda$$

$$:\alpha'\gamma\,(\lambda-1)-\alpha\gamma'\,(\mu-1)\,:\,\alpha'\beta\,\lambda-\alpha\beta'\mu \quad . \quad . \quad (3)$$

Cor. The equation to a line being of the form $A/yz+B/zx+C/xy=0$, it follows that to a straight line in general corresponds a cubic in λ and μ of the second degree at most in either λ or μ. If, however, 4 and 5 are chosen so that $\beta/\beta'=\gamma/\gamma'$,

i.e., if they are in a line with 1, then to a straight line corresponds a quadratic equation in λ and μ. For, if $\beta/\beta' = \gamma/\gamma'$

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \mathrm{L} : \mathrm{M} : \mathrm{N}$$

where L, M, N are linear functions of λ and μ; and $\therefore$ to $\Sigma \mathrm{A}/yz = 0$ corresponds

$$\mathrm{AMN} + \mathrm{BNL} + \mathrm{CLM} = 0.$$

Similarly if 2, 3, 4 say are collinear, and also 1, 2, 5 then the equations become

$$\frac{\mathrm{P}13}{\mathrm{P}14} = c\lambda\,; \quad \frac{\mathrm{P}13}{\mathrm{P}23} = c'\mu\,;$$

in which $c = 23/24$; $c' = 15/25$. In such a case to a linear equation in x and y corresponds a linear equation in λ and μ and inversely. We shall in general assume that no three points in question are collinear.

§ 7. A number of interesting geometrical theorems may be more easily deduced by taking particular algebraic relations connecting λ and μ. In what follows, ordinary cartesian co-ordinates are used, and by (Pab) is meant the determinant

$$\begin{vmatrix} x & y & 1 \\ x_a & y_a & 1 \\ x_b & y_b & 1 \end{vmatrix}$$

Ex. 1. To $\lambda = c$ a constant corresponds the conic section given by

$$(\mathrm{P}13)\ (\mathrm{P}24) = c\ (\mathrm{P}14)\ (\mathrm{P}23.)$$

Ex. 2. To the relation

$$\mathrm{A}\lambda + \mathrm{B}\mu = 0$$

corresponds likewise a conic through the points 1, 2, 4, 5 but not in general through 3.

For $\qquad \mathrm{P}(1234) \,/\, \mathrm{P}(1235) = \mathrm{P}(1254).$

$$\therefore \mathrm{P}(1254) = -\mathrm{B}\,/\,\mathrm{A} = \text{constant}$$

Ex. 3. To $\qquad \mathrm{A}\lambda + \mathrm{B}\mu + \mathrm{C} = 0$

corresponds $\quad \mathrm{A} \times \mathrm{P}(1234) + \mathrm{B} \times \mathrm{P}(1235) + \mathrm{C} = 0,$

i.e., $\mathrm{A}(\mathrm{P}13)(\mathrm{P}15)(\mathrm{P}24) + \mathrm{B}(\mathrm{P}13)(\mathrm{P}14)(\mathrm{P}25) + \mathrm{C}(\mathrm{P}14)(\mathrm{P}15)(\mathrm{P}23) = 0$

This equation in general represents a cubic curve possessing a double point at 1, and ordinary points at 2, 3, 4, 5.

Now the datum of a double point is equivalent to three conditions, and each ordinary point is given by one condition. There are therefore a twofold infinity of cubics possessing a double point at 1 and through 2, 3, 4, 5. Also the equation in λ and μ contains two arbitrary constants. We have, therefore, the following theorem suggested.

"Take a cubic with double point at 1, and let 2, 3, 4, 5 be any four fixed points on it. Let π_1 denote P(1234), and κ_1 denote P(1235), P being any other point on the cubic. There is a linear relation connecting π_1 and κ_1, and $(\pi_1 \pi_2 \pi_3 \pi_4) = (\kappa_1 \kappa_1 \kappa_3 \kappa_4)$."

For confirmation see Salmon's Higher Plane Curves, § 163.

Particular cases arise when 2 and 3 are the circular points at infinity.

But if $5(1234) = \alpha$, and $4(1235) = \beta$; and if $A\alpha + B\beta + C = 0$, then the curve given by $A\lambda + B\mu + C = 0$ is a degenerate cubic consisting of a conic and a straight line through 1.

Ex. 4. To the relation

$$A\lambda\mu + B\lambda + C\mu + D = 0$$

corresponds

$$\begin{aligned} &A(P13)^2(P24)(P25) + B(P13)(P23)(P15)(P24) \\ &+ C(P13)(P23)(P14)(P25) + D(P23)^2(P14)(P15) = 0 \end{aligned}$$

This equation represents a quartic curve in general, possessing double points at 1, 2, 3, and ordinary points at 4, 5.

There are three arbitrary constants in the $\lambda - \mu$ equation. But only a threefold infinity of quartics are possible possessing nodes at three given points and through other two points. We have therefore the following theorem suggested.

"Take a tri-nodal quartic with nodes at 1, 2, 3. Let 4 and 5 be any two fixed points on it, and P an arbitrary point on the curve. Then $(\pi_1 \pi_2 \pi_3 \pi_4) = (\kappa_1 \kappa_2 \kappa_3 \kappa_4)$."

There are a variety of degenerate cases. For example if ABCD are such that $A\alpha\beta + B\alpha + C\beta + D = 0$, then the quartic reduces to the base conic 12345, and a second conic through 1, 2, 3, (Degenerate cases may also arise should any of the base points be collinear).

Any relation $f(\lambda, \mu)=0$ will furnish a degenerate curve when $f(\alpha, \beta)=0$.

Ex. 5. If a bilinear equation is given connecting P(1234) and P(5678) the locus of P is in general a quartic; but in a large variety of cases the curve is of lower order.

(*a.*) If P(1342)/P(1562) = λ, a constant, the locus of P is a cubic

$$(P14)(P32)(P56)-\lambda(P34)(P16)(P52)=0$$

passing through 1, 2, 3, 4, 5, 6; and through the intersection 7 of 14 and 52; 8 of 32 and 16; 9 of 56 and 34. (Figure 8.)

The nine points thus obtained form nine associated points of a pencil if cubics in triads upon two systems of three lines; and one obtains (*v.* Salmon's Higher Plane Curves) the most general form of the cubic, from which its more elementary properties are generally deduced. It will be noted that if 3, 4; and 1, 2, are given on a fixed cubic, the points 5 and 6 are uniquely determined by the collinearities:—

$$\begin{matrix} 1\,4\,7 \\ 2\,7\,5 \end{matrix} \quad \text{and} \quad \begin{matrix} 2\,3\,8 \\ 1\,8\,6. \end{matrix}$$

Also 34 and 56 cut on the cubic.

The same cubic could be obtained by a variety of such equations, *e.g.*, P(5164)/P(5324) = μ, corresponding to

$$\begin{matrix} 6\,5\,9 \\ 9\,4\,3 \end{matrix} \quad \text{and} \quad \begin{matrix} 1\,4\,7 \\ 7\,5\,2. \end{matrix}$$

Since all the relations are algebraic, a bilinear equation, $\lambda=\mu$, connecting λ and μ is suggested for the same cubic.

(*b.*) P(2134)/P(2536) = λ gives

$$(P14)(P26)(P53)-\lambda(P56)(P24)(P13)=0,$$

a cubic through 1, 2, 3, 4, 5, 6 and through the intersections 7, 8, 9 of 14 and 56; 26 and 13; 53 and 24. (Figure 9.)

These again form nine associated points. They may also be found in any given cubic as follows.

Take the base points 1, 2, 5; and the point 7. Form the Steinerian hexagon

$$\begin{matrix} & 7\,1\,4 \\ & 4\,2\,9 \\ & 9\,5\,3 \\ & 3\,1\,8 \\ & 8\,2\,6 \\ \text{and} \therefore & 6\,5\,7 \end{matrix}$$

Then P(2134)/P(2536) = constant for all points P on the cubic.
Similiar base points for the same configuration are

169 : 237 : 346 : 458 : 789.

Thus start with 1, ($\therefore$ 7438 excluded). The other two base points must come from 2569. Take 6, thereby excluding 5728, *i.e.*, excluding in addition 5 and 2, and leaving 9; $\therefore$ 169 as possible base points

Thus

4 1 7
7 6 5
5 9 3
3 1 8
8 6 2
2 9 4;

and P(6137)/P(6932) = μ a constant.

Ex. 6. Consider the cross ratios

1(2345): 2(3451); ......; 5(1234).

Equate the product of two or more of these to a constant. Fix four of the points, when the fifth traces out a curve which is at most of the fourth degree, and is generally, but not always, unicursal.

(A) *e.g.* Let 1(2345) . 2(3451) . 3(4512) = constant.

(i) Put P for 5

$\therefore$ $(P23)^2/(P12)(P24) =$ constant

$\therefore$ the locus of P is a pair of lines.

(ii) Put P for 4. The locus is again a degenerate conic.

(iii) Put P for 3

$\therefore$ (P15)(P25)(P23)/(P12)(P24)(P13) = constant.

The locus is a cubic with double point at 2; through 1 and 3; through the intersection of 15 and 24, and of 25 and 13; and tangent to 15 at 1.

(iv) Put P for 2

$\therefore$ $(P14)^2(P35)^2/(P13)(P15)(P23)(P45) =$ constant.

The locus is a quartic with double points at 1, 3, 5, through 4, but not through 2; 23 is tangent where 14 again cuts the curve; 45 is tangent at 4.

(v) Put P for 1 when the locus is a pair of lines through 2.

(B) Equate the product of the five ratios to a constant and put P for 5, say.

$$\therefore\ P13\,.\,P23\,.\,P24/P12\,.\,P34\,.\,P14 = \text{constant}.$$

This is a cubic through 1, 2, 3, 4: 13 is tangent at 1, 34 is tangent at 3, 42 is tangent at 4, and 21 is tangent at 2. Also 23 and 14 intersect on the curve.

The quadrilateral 1234 is therefore both inscribed and circumscribed to the cubic. A cubic curve with no double point has always a limited number of such quadrilaterals.

Ex. 7. $$P(1234) \times P(1278) = \text{constant}$$

gives a quartic with double points at 1 and 2.

$$P(1234) \times P(1678) = \text{constant}$$

furnishes a quartic with a double point at 1.

$$P(1234)\ P(5678) = \text{constant}$$

furnishes a quartic which does not in general possess a double point.

On the Geometry of the Conic and Triangle.

By JOHN MILLER.

(*Read March 13th. Received, June 1st, 1908.*)

§ 1.

In the *Proceedings* of 1905–6 Mr Pinkerton gave an extension of the nine point circle to a nine point conic. This raises the question of the extension of the geometry of the circle and triangle to that of the conic and triangle. If a triangle with its associated system of lines and circles be orthogonally projected on a second plane we have a triangle with an associated system of lines and homothetic ellipses. Pairs of perpendicular lines are projected into lines parallel to pairs of conjugate diameters. Such lines will be called, for shortness, in the sequel, conjugate lines. In any relation between lengths of lines, these lengths will be replaced by their ratios to the lengths of the parallel radii of one of the homothetic ellipses.

The question now arises whether this geometry of the triangle holds when instead of this system of homothetic ellipses we have hyperbolas. On projecting a circle into a hyperbola, pairs of perpendicular lines do not project into pairs of conjugate lines, for the lines perpendicular to a given line project into concurrent lines. Further, the circles of a plane are not all projected into hyperbolas, much less into homothetic hyperbolas.

Nevertheless, the corresponding geometry exists. The present paper is an attempt to indicate by geometric proofs the point of continuity with the proofs for the geometry of the triangle and circle. The amount of material is very large so that only some excerpts are given and the proofs of some theorems which the extension suggests are omitted to shorten the paper.

A system of homothetic conics has two real or imaginary common points at infinity. A hyperbola and its conjugates will be considered homothetic. Thus circles have in common the two circular points at infinity. Each conic, therefore, of the system is determined by three points in the finite part of the plane. This is the first principle on which the geometry depends.

We have next the theorem that the ratio of the products of the segments of two chords or secants of a conic through any point is equal to the ratio of the squares on parallel radii. For the circle this ratio is unity, and we have similar triangles bringing in equality of angles. Also, since all the radii of a circle are equal, angles in the same segment are equal. The proofs of the geometry of the circle and triangle generally depend on using relations between angles. It will be seen that in the extension to conics the proofs usually involve the theorem concerning the segments of two chords or secants.*

§ 2.

"ORTHOCENTRE." NINE POINT CONIC.

Let O (Fig. 10) be the centre of a circumconic of a triangle ABC; U, V, W the mid points of the sides; AD, BE, CF parallel to OU, OV, OW; a_1^2, b_1^2, c_1^2 the squares of the radii parallel to a, b, c.

Since BE, CE; CF, BF are pairs of conjugates, a homothetic conic with BC as diameter passes through E and F.

$$\therefore \frac{\mathrm{AF}\,.\,c}{\mathrm{AE}\,.\,b}=\frac{c_1^2}{b_1^2}.\quad \text{Similarly } \frac{\mathrm{BD}\,.\,a}{\mathrm{BF}\,.\,c}=\frac{a_1^2}{c_1^2} \text{ and } \frac{\mathrm{CE}\,.\,b}{\mathrm{CD}\,.\,a}=\frac{b_1^2}{a_1^2}.$$

Hence by Ceva's theorem AD, BE, CF intersect in a point H. From the first equation

$$\frac{(c-\mathrm{BF})c}{c_1^2}=\frac{(b-\mathrm{CE})b}{b_1^2}$$

or

$$\frac{b\,.\,\mathrm{CE}}{b_1^2}-\frac{c\,.\,\mathrm{BF}}{c_1^2}=\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2}.$$

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* I am indebted to *Dr Muirhead* for reference to a paper by *L. Ripert* (*La Dualité et l'Homographie dans le Triangle et le Tétrahèdre*, Paris, Gauthier-Villars et Fils, 1898.) In this short generalising paper M. Ripert has anticipated some of my fundamental ideas. His work is, however, very different, being mainly an application of barycentric co-ordinates. I may state, that in the detailed working out of results, I often used proofs by trilinear and barycentric co-ordinates, but discarded them for proofs involving the direct application of the principles already mentioned.

From the second by transposing and adding

$$\frac{b\,.\,\mathrm{CE}}{b_1^2}+\frac{c\,.\,\mathrm{BF}}{c_1^2}=\frac{a^2}{a'^2}.$$

$$\therefore\ \frac{2b\,.\,\mathrm{CE}}{b_1^2}=\frac{2a\,.\,\mathrm{CD}}{a_1^2}=\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2},\ \text{etc.}$$

We give a second simple proof.

Let XOY′ parallel to c cut a in X and b in Y′;
X′OZ parallel to b cut a in X′ and c in Z;
Z′OY parallel to a cut c in Z′ and b in Y. (Fig. 10.)

Since $\frac{\mathrm{XU}}{\mathrm{UX'}}=\frac{\mathrm{BD}}{\mathrm{DC}}$ it is to be proved that

$$\frac{\mathrm{XU}}{\mathrm{UX'}}\cdot\frac{\mathrm{YV}}{\mathrm{VY'}}\cdot\frac{\mathrm{ZW}}{\mathrm{WZ'}}=1.$$

Now $$\frac{\mathrm{AZ'}}{c}=\frac{\mathrm{AY}}{b}\ \text{or}\ \frac{\frac{1}{2}c+\mathrm{WZ'}}{c}=\frac{\frac{1}{2}b+\mathrm{VY}}{b}.$$

$$\therefore\ \frac{\mathrm{WZ'}}{\mathrm{VY}}=\frac{c}{b}.\quad\therefore\ \text{etc.}$$

The proof for the nine point conic now proceeds as in Mr Pinkerton's paper.

Let (Fig. 10) AO and AD meet the circumconic again in O′ and D′. O′D′ is parallel to UD. If OU meet O′D′ in U′, OU′ $=\frac{1}{2}$AD′. But OU $=\frac{1}{2}$AH. $\therefore$ HD = DD′.

Let p^2 be the square of the radius parallel to AD.

$$\frac{\mathrm{AD}\,.\,\mathrm{DD'}}{p^2}=\frac{\mathrm{AD}\,.\,\mathrm{HD}}{p^2}=\frac{\mathrm{BD}\,.\,\mathrm{DC}}{a_1^2}=\left(\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}-\frac{c^2}{c_1^2}\right)\left(\frac{a^2}{a_1^2}-\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}\right)\div\frac{2a^2}{a_1^2}.$$

Also a homothetic conic with CH as diameter passes through D, E.

$$\therefore\ \frac{2\mathrm{AH}\,.\,\mathrm{AD}}{p^2}=\frac{2b\,.\,\mathrm{AE}}{b_1^2}=\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}.$$

$$\therefore\ \frac{2\mathrm{AD}^2}{p^2}-\frac{2\mathrm{AD}.\,\mathrm{HD}}{p^2}=\frac{b^2}{b_1^2}+\frac{c^2}{c_1^2}-\frac{a^2}{a_1^2}.$$

$$\therefore\ \frac{4a^2AD^2}{p^2a_1^2} = S^2 = s\left(s - \frac{a}{a_1}\right)\left(s - \frac{b}{b_1}\right)\left(s - \frac{c}{c_1}\right) \text{ where } 2s = \frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1}.$$

It may be noted that if the conic be a hyperbola the results are still real since a radius enters by its square only. These results show that extended trigonometrical or hyperbolic sines or cosines might be used with advantage. An angle will have to be supposed given only when the directions of both arms are given. If a homothetic conic be drawn with the intersection as centre and from the end of one arm the conjugate to the second arm be drawn, the sine will be the ratio of this ordinate to the parallel radius and the cosine will be the ratio of the abscissa to the parallel radius.

§ 3.

"WALLACE LINE."

Let P be a point on the circumconic and PQ, PR, PS conjugate to BC, CA, AB. (Fig. 11).

Since PQ, CQ are conjugate and PR, CR are conjugate, the homothetic conic on PC as diameter passes through Q, R. Let PQ, PR, PS cut the circumconic in Q_1, R_1, S_1. The circumconic, the conic PCQR and the pair of lines PQ, CR have a common chord CP; therefore their other three chords meet in a point. The common chord of the two homothetic chords is at infinity and therefore QR, AQ_1 are parallel. Similarly taking the pair of lines CQ, PR we find that QR and BR_1 are parallel. By considering the quadrilateral PRAS we find that RS and BR_1 are parallel and therefore Q, R, S are collinear.

If H_1 be taken on QP such that $QH_1 = Q_1Q$, H_1 is the "orthocentre" of the triangle PBC and $PH_1 = 2OU = AH$. Let the "Wallace line" meet AD in X (Fig. 11); QQ_1AX is a parallelogram and $Q_1Q = AX$. Therefore $QP = HX$ and so the "Wallace line" bisects PH in M which is a point on the nine point conic.

Let N be the centre of the nine point conic and G the centroid. Let GP cut OM in G_1 and draw GG_2 parallel to NM to cut OM in G_2. (Fig. 12).

Since $OG = \frac{2}{3}ON$, $GG_2 = \frac{2}{3}NM$,

$$\frac{PG_1}{G_1G} = \frac{OP}{GG_2} = \frac{2NM}{\frac{2}{3}NM} = \frac{3}{1}.$$

$\therefore$ **G_1 is the centre of mean position of A, B, C, P.**

Also
$$\frac{OG_1}{G_1G_2} = \frac{3}{1} \text{ and } OG_2 = \frac{2}{3}OM.$$

$$\therefore\ OG_1 = 3(OG_2 - OG_1) = 2OM - 3OG_1 = \tfrac{1}{2}OM.$$

$\therefore$ M is got by joining the centre of the circumconic to the centre of mean position of the four points and producing this line its own length. The symmetry shows that the four "Wallace lines" obtained by taking three of the points as vertices and the fourth as the point from which the conjugates are drawn, all meet in a point.

Any transversal of a triangle can be regarded in an infinite number of ways as a "Wallace line." Draw (Fig. 11) QP arbitrarily and let the parallel to QP through A cut the transversal in X. Bisect QX in M. We have then five points U, V, W, D, M on the nine point conic. These determine it and the homothetic circumconic.

The locus of P for a given transversal is a straight line. The locus of the "orthocentre" H is a hyperbola through A, B, C with asymptotes parallel to QRS and the locus of P. The locus of the centre O is a hyperbola passing through U, V, W and with asymptotes parallel to the same lines.

If P, A, B, C be given the envelope of the "Wallace line" of P is a conic touching the sides AB, BC, CA. The loci of M, N, H are conics, etc.

§ 4.

Conics Touching the Sides.

Four conics homothetic to the circumconic can be drawn to touch the sides unless the circumconic is a hyperbola and the

vertices are not all on the same branch; in this case the tangent conics are imaginary. Draw the diameters of the circumconic conjugate to the sides BC, CA, AB and draw tangents at their ends. These form in all eight triangles similar to ABC, but four are equal to the remaining four. We give a figure (Fig. 13) for the ellipse only. In this figure $A_1B_1C_1$ and the conic correspond to ABC and the inscribed ellipse: $A_1B_2C_2$ and the conic correspond to ABC and the ellipse touching BC externally. The centres I, I_1, I_2, I_3 can now be easily found for O say lies with respect to A_1, B_2, C_2 as I_1 with respect to A, B, C.

A, I, I_1 are collinear and I_2, A, I_3.

OA_1 and OA_2 are conjugate diameters. Therefore AII_1 and I_2AI_3 are conjugate or ABC is the "pedal" triangle of $I_1I_2I_3$ and I the "orthocentre." The circumconic of ABC is the nine point conic of $I_1I_2I_3$ and the points K and L say where it meets I_2I_3, II_1 again are the mid points of I_2I_3 and II_1.

Since KA and LA are conjugate, KL is a diameter. Also since the mid points of the diagonals of the quadrilateral BI_1CI are collinear, KL bisects BC in U and is therefore parallel to the radii r, r_1, r_2, r_3 drawn from I, I_1, I_2, I_3 to the points of contact of the conics with BC.

If $KL = 2R$, $r_2 + r_3 = 2KU$ and $r_1 - r = 2UL$. Therefore if r, r_1, r_2, r_3, R be any parallel radii

$$r_1 + r_2 + r_3 - r = 4R.$$

The preceding results give a method of inscribing in a conic a triangle whose sides will be parallel to those of a given triangle. Draw three chords parallel to the sides and find their diameters. Through three of the ends of the diameters draw tangents giving a triangle circumscribed to the conic and similar to the given triangle. Through the point of contact of a side draw a parallel to the line joining the opposite vertex to the centre. The three lines thus drawn meet the conic again in the vertices of the required triangle.

If the inscribed ellipse or corresponding hyperbola touch AB in N and AC in M, then $\dfrac{AM^2}{b_1^2} = \dfrac{AN^2}{c_1^2}$, etc.

Hence $$\frac{AM}{b_1}=\frac{AN}{c_1}=\tfrac{1}{2}\left(\frac{b}{b_1}+\frac{c}{c_1}-\frac{a}{a_1}\right)=s-\frac{a}{a_1}.$$

Similarly we can find the other segments.

Let I be the centre of the inscribed ellipse or corresponding hyperbola and let r and R be the radius of the inconic along AI and of the circumconic parallel to AI.

$$\frac{(AI-r)(AI+r)}{R^2}=\frac{AN^2}{c_1^{\,2}}=\left(s-\frac{a}{a_1}\right)^2$$

$$\therefore\ \frac{AI^2}{R^2}=\left(s-\frac{a}{a_1}\right)^2+\frac{r^2}{R^2}.$$

Similarly $$\frac{AI_1^{\,2}}{R^2}=s^2+\frac{r_1^{\,2}}{R^2}.$$

Also $$\frac{AI}{AI_1}=\frac{s-\dfrac{a}{a_1}}{s}\quad\text{and}\quad\frac{r_1}{r}=\frac{4R}{r}+1-\frac{r_2}{r}-\frac{r_3}{r}$$

$$=\frac{4R}{r}+1-\frac{s}{s-\dfrac{b}{b_1}}-\frac{s}{s-\dfrac{c}{c_1}}.$$

From these the ratio of similarity $\frac{r}{R}$ is found to be

$$\frac{4\left(s-\dfrac{a}{a_1}\right)\left(s-\dfrac{b}{b_1}\right)\left(s-\dfrac{c}{c_1}\right)}{\dfrac{a\,b\,c}{a_1b_1c_1}}.$$

§ 5.

"ISOGONAL CONJUGATES."

If AO, AO_1 are two lines such that when the conjugates ON, O_1N_1 are drawn to AB and the conjugates OM, OM_1 to AC

$$\frac{ON \cdot O_1N_1}{r^2} = \frac{OM \cdot O_1M_1}{q^2}$$

where p, q, r are the radii conjugate to a, b, c, then AO, AO_1 will be called "isogonally conjugate."

If three lines from the vertices are concurrent their "isogonal conjugates" are concurrent. Let O, O_1 be the points of concurrency called "isogonal conjugate" points. If the circumconic is a hyperbola and A is on a different branch from B and C, O and O_1 are on the same side of AB or AC but on different sides of BC.

If OL, O_1L_1 be conjugate to BC the six points L, L_1, M, M_1, N, N_1 lie on a homothetic conic whose centre is P the mid point of OO_1. Take the homothetic conic with centre P and passing through L. It will pass through L_1 and by a little indirect work can be proved to pass through M, M_1, N, N_1.

If the conjugates AD, BE, CF to the sides meet the circumconic in D_1, E_1, F_1 then

$$\frac{AH \cdot HD_1}{BH \cdot HE_1} = \frac{p^2}{q^2}.$$

But $AH = 2OU$, $BH = 2OV$, $HD_1 = 2HD$, $HE_1 = 2HE$; $\therefore$ the circumcentre O and the "orthocentre" H are "isogonally conjugate."

§ 6.

"Antiparallels."

A line MN parallel to the tangent at A is "antiparallel" to BC. Let MN cut AC in M and AB in N. Consider the homothetic conic through B, C, M. This, the circumconic and the lines AB, AC have the common chord BC. Their other three chords then are concurrent. Hence the second chord of the conic BCM and the lines AB, AC is parallel to the tangent at A. This conic then passes through N.

The sides of the "pedal" triangle are therefore parallel to the tangents at the vertices.

Let P, Q, R be the poles of the sides with regard to the circumconic. Draw the "antiparallel" to BC through P meeting AB in N and AC in M.

The triangles PCM, QCA are similar and the triangles NBP, ABR.

$$\therefore \frac{PM}{PC}=\frac{AQ}{CQ} \text{ and } \frac{NP}{BP}=\frac{RA}{RB}$$

$$\therefore \frac{NP}{PM}=\frac{RA}{AQ}\cdot\frac{QC}{CP}\cdot\frac{PB}{BR}=1$$

since AP, BQ, CR are concurrent.

$\therefore$ NP = PM and the bisectors AP, BQ, CR of the "antiparallels" meet in the "symmedian" centre K. The locus of the "symmedian" centre K of the triangle ABC for all conics passing through the four points Z, A, B, C is the polar of the point Z with respect to the triangle ABC.

§ 7.

"Cosine" Conic

Through the "symmedian point" K let the antiparallel YZ_1 to BC be drawn cutting AC in Y and AB in Z_1 (Fig. 14.) Similarly let the antiparallels ZX_1, XY_1 to CA and AB cut BC in X and X_1, AB in Z and AC in Y_1.

Then $\qquad ZK = KX_1,\ Z_1K = KY,\ XK = KY_1.$

$\therefore$ ZY_1 is parallel to BC, Z_1X to AC and X_1Y to AB.

$$\therefore \frac{AZ}{AY_1}=\frac{c}{b},\ \frac{BX}{BZ_1}=\frac{a}{c},\ \frac{CY}{CX_1}=\frac{b}{a}.$$

$$\therefore \frac{AZ\ .\ AZ_1\ .\ BX\ .\ BX_1\ .\ CY\ .\ CY_1}{AY_1\ .\ AY\ .\ CX_1\ .\ CX\ .\ BZ_1\ .\ BZ}=\frac{AZ_1\ .\ BX_1\ .\ CY_1}{AY\ .\ CX\ .\ BZ}.$$

But if AD, BE, CF are conjugate to the sides, $\frac{AZ_1}{AY}=\frac{AF}{AE}$, etc.

$$\therefore \frac{AZ_1\ .\ BX_1\ .\ CY_1}{AY\ .\ CX\ .\ BZ}=\frac{AF\ .\ BD\ .\ CE}{AE\ .\ BF\ .\ CD}=1.$$

Therefore, by Carnot's theorem the six points X, X_1, Y, Y_1, Z, Z_1, lie on a conic.

Let a_2, b_2, c_2 be the radii of this conic parallel to the sides.

Then $\dfrac{AZ\ .\ AZ_1}{AY_1\ .\ AY}=\dfrac{c_2^2}{b_2^2}$. But $\dfrac{AZ\ .\ AZ_1}{AY_1,\ AY}=\dfrac{c\ .\ AF}{b\ .\ AE}=\dfrac{c_1^2}{b_1^2}$.

$$\therefore\ \frac{c_2^2}{c_1^2}=\frac{b_2^2}{b_1^2}=\frac{a_2^2}{a_1^2},$$

and the conic is homothetic with the circumconic and the centre is K.

X_1Y_1, Y_1Z_1, Z_1X_1 as also XZ, YX, ZY are conjugate to BC, CA, AB. Therefore through the vertices of a triangle two triangles can be drawn with sides conjugate in reverse order to those of the first triangle and both have for "symmedian" or "cosine" centre the circumcentre of the triangle, etc.

$$\frac{BX}{a}+\frac{XX_1}{a}+\frac{X_1C}{a}=1,$$

$\therefore$ since Z_1X, X_1Y are equal and parallel to Y_1Y, Z_1Z

$$\frac{XX_1}{a}+\frac{YY_1}{b}+\frac{ZZ_1}{c}=1.$$

Also
$$\frac{CX_1(CX_1+X_1X)}{a_1^2}=\frac{CY(CY+YY_1)}{b_1^2}.$$

$$\therefore\ \frac{\dfrac{XX_1}{a}+\dfrac{ZZ_1}{c}}{\dfrac{a_1^2}{a^2}}=\frac{\dfrac{ZZ_1}{c}+\dfrac{YY_1}{b}}{\dfrac{b_1^2}{b^2}}.$$

Similarly
$$\frac{\dfrac{YY_1}{b}+\dfrac{XX_1}{a}}{\dfrac{a_1^2}{a^2}}=\frac{\dfrac{YY_1}{b}+\dfrac{ZZ_1}{c}}{\dfrac{c_1^2}{c^2}}.$$

From these

$$\frac{\dfrac{XX_1}{a}}{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}-\dfrac{a^2}{a_1^2}}=\frac{\dfrac{YY_1}{b}}{\dfrac{c^2}{c_1^2}+\dfrac{a^2}{a_1^2}-\dfrac{b^2}{b_1^2}}=\frac{\dfrac{ZZ_1}{c}}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}-\dfrac{c^2}{c_1^2}}=\frac{1}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}.$$

Let A_2, B_2, C_2 be the mid points of XX_1, YY_1, ZZ_1.
$KB_2=\frac{1}{2}XY$, and XY is parallel to BE.

$$\therefore \frac{XY}{BE}=\frac{CX}{a}=\frac{CX_1}{a}+\frac{X_1X}{a}=\frac{XX_1}{a}+\frac{ZZ_1}{c}=\frac{\dfrac{2b^2}{b_1^2}}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}.$$

$$\therefore KB_2=\frac{\dfrac{b^2\,.\,BE}{b_1^2}}{\dfrac{a^2}{a_1^2}+\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}}=\frac{bqS}{2b_1\Sigma\dfrac{a^2}{a_1^2}}.$$

$$\therefore \frac{\dfrac{KA_2}{p}}{\dfrac{a}{a_1}}=\frac{\dfrac{KB_2}{q}}{\dfrac{b}{b_1}}=\frac{\dfrac{KC_2}{r}}{\dfrac{c}{c_1}}=\frac{S}{2\Sigma\dfrac{a^2}{a_1^2}}.$$

If U is the mid point of BC and UE_1 is conjugate to CA then
$$KB_2\,.\,UE_1=\tfrac{1}{2}KB_2\,.\,BE=\frac{q^2S}{8\Sigma\dfrac{a^2}{a_1^2}}.$$

$$\therefore \frac{KB_2\,.\,UE_1}{q^2}=\frac{KC_2\,.\,UF_1}{r^2}.$$

$\therefore$ G and K are "isogonally conjugate."

If T is the pole of XX_1 with regard to the "cosine" conic, the triangles BOC, X_1TX are similar.

$$\therefore \frac{TA_2}{OU}=\frac{XX_1}{a}=\frac{\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}-\dfrac{a^2}{a_1^2}}{\Sigma\dfrac{a^2}{a_1^2}},$$

$$\therefore\ \frac{\mathrm{TA}_2}{p}=\frac{\dfrac{a}{a_1}\left(\dfrac{b^2}{b_1^2}+\dfrac{c^2}{c_1^2}-\dfrac{a^2}{a_1^2}\right)^2}{2\mathrm{S}\Sigma\dfrac{a^2}{a_1^2}}.$$

$$\mathrm{KT}=\mathrm{KA}_2+\mathrm{A}_2\mathrm{T}=\frac{2p\dfrac{a^2b^2c^2}{a_1^2b_1^2c_1^2}}{\mathrm{S}\Sigma\dfrac{a^2}{a_1^2}}.$$

KT . $\mathrm{KA}_2=p_1^2$ where p_1 is the radius of the "cosine" conic parallel to p.

$$\therefore \text{ the ratio of similarity } \frac{p_1}{p}=\frac{\dfrac{abc}{a_1b_1c_1}}{\Sigma\dfrac{a^2}{a_1^2}}.$$

Enough has been done to show the extensions and methods of proof. Considerations of space also prevent me adding investigations I have made on the extensions of Lemoine's, Tucker's, and Taylor's circles, on the Brocard points, on Inversion, etc.

Notes on the Apollonian problem and the allied theory. Part II.

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In a paper under the above title in Vol. XXIV. of the *Proceedings* it is shown that in a certain system of co-ordinates, the equation of the first degree represents a circle orthogonal to a fixed circle. It follows that any purely graphical theorem regarding right lines in a plane can be extended to orthogonals to a circle. This may be seen otherwise by projecting the figure of right lines on a sphere, the right lines thus becoming circles orthogonal to a circle on the sphere; and then inverting the sphere into the original plane. The geometrical method shows that the extension may also be applied to theorems involving one circle as well as right lines, the circle remaining unchanged, while the lines become orthogonals to a circle; the Pole and Polar Theorem, Pascal's and Brianchon's Theorems are examples. But plane figures involving more than one circle cannot in general be transformed in this way. We cannot, for instance, deduce the construction for a circle touching three great circles on a sphere from the known construction for a circle touching three lines in a plane; nor the Gergonne construction for circles on a sphere from the corresponding method in a plane.

The following pages contain proofs of the extended forms of the Pole and Polar Theorem, Pascal's Theorem, and Gergonne's construction. The theorems in their ordinary forms are not assumed, and the method of proof is as applicable to circles on a sphere as in a plane.

A construction is given for the Apollonian problem, perfectly analogous to the usual construction for a plane or spherical triangle. It may, of course, be got at once by a real or imaginary inversion from the latter case.

I hope to continue the notes in a later communication, and to deduce from them a geometrical theory of conics having double contact with a conic.

2. In the paper cited above, certain conventions were used, which will be adopted here, and it may be convenient for the reader to have a brief account of the chief of them before him :

(*a*) The *radius* of a circle is supposed to have algebraic sign, so that if the square of the radius is a^2, the circle $-a$ is distinguished from the circle a, although both contain the same points, real and imaginary.

(*b*) The *angle* Θ between two circles a, b, is defined by the relation

$$d^2 = a^2 + b^2 - 2ab \cos \Theta,$$

where d is the distance between the centres.

(*c*) Two circles a, b *touch* if $\Theta = \mathrm{O}$, or $d^2 = (a-b)^2$.

(*d*) The *centre of similitude* of two circles a, b, is that point S in the line of centres A, B, for which SA : SB $= a : b$.

It follows that when a, b touch, the point of contact is the centre of similitude.

(*e*) The signs of the radii of two inverse circles are so related that the centre of the circle of inversion is their centre of similitude.

(*f*) Subject to these conventions, it was shown that the angle between two circles is *equal* to the angle between their inverses; that the circle of inversion with radius $-k$ is inverse to the same circle with radius k; and that consequently a circle inverse to itself, *i.e.*, an orthogonal to the circle of inversion, cuts two inverse circles at equal angles; and conversely that a circle cutting two inverse circles at equal angles is orthogonal to the circle of inversion ; but that the circle of inversion itself cuts two inverse circles at *supplementary* angles.

(*g*) It will, however, be convenient, and should hardly be misleading, to refer to the circle of inversion of a, b as their bisecting circle, or the *bisector* of the angle between them. It is the circle coaxal with a, b, and having its centre at the centre of similitude.

3. We take a fixed circle O as the base to which orthogonal circles are drawn. Such circles will be called *orthogonals* simply. Two orthogonals intersect in a pair of *inverse points.*

In general, the centre of O is supposed to be a real point, but the square of its radius is not necessarily positive.

Graphically, a system of orthogonals to a circle is simply a system of circles, the common chord of every pair of which passes through a fixed point.

This holds on a sphere also; in this case the planes of orthogonals to a circle C pass through a fixed point, the pole of the plane of C; thus great circles are orthogonal to the imaginary circle in which the plane at infinity cuts the sphere.

4. The ordinary centre of a circle is the point through which all right lines orthogonal to the circle pass.

By analogy, we define the *centre as to O* of a circle C as the pair of points to which all common orthogonals to C and O pass. These are the limiting points of the coaxal system to which C and O belong, and are inverse points with respect to C and to O alike.

All circles cutting O in the same two points have the same centre as to O.

The orthogonal through the centres as to O of two circles may be called their *line of centres as to O.*

If two circles touch, their line of centres as to O goes through their point of contact, for the two limiting points coincide there.

5. The *common chord*, or *radical axis, as to O*, of two circles is the orthogonal through their common points, or coaxal with them.

It is not a determinate circle when the given circles are themselves orthogonals.

The line of centres as to O of the circles being orthogonal to both circles, is orthogonal to their common chord.

The radical axes as to O of three circles are coaxal, for each is orthogonal to two circles, viz., O and the orthogonal circle of the three circles. The common point pair of these radical axes is the *radical centre as to O* of the three circles. It is the centre as to O of their orthogonal circle.

6. The centres as to O of a system of coaxal circles lie on an orthogonal, viz., the orthogonal through their limiting points.

7. Every right line making equal angles with two circles *a*, *b*, being orthogonal to the bisecting circle, § 2 (*f*), goes through the

centre of similitude, and conversely. Similarly every orthogonal making equal angles with two circles a, b, is orthogonal to two circles, O and the bisecting circle, and therefore goes through a fixed point pair, the *centre of similitude as to O* of a, b; and conversely every circle through this point pair cuts a, b, at equal angles.

The centre of similitude as to O of a, b, is the intersection of the two orthogonals which touch them both.

8. The line of centres as to O of two circles, being orthogonal to both, makes both equal and supplementary angles with them, and therefore goes through both centres of similitude as to O.

9. By § 2 (b) every circle which cuts a point circle, *i.e.*, a circle of zero radius, at a finite angle passes through the point.

Also by (d), (g) if two circles a, b, touch, their bisector is the point circle with centre at the point of contact.

Hence every circle cutting two touching circles a, b, at equal angles, goes through their point of contact, and conversely (*cf.* last sentence of § 4).

10. The three centres of similitude as to O of three circles a, b, c, taken in pairs, lie on an orthogonal, the *axis of similitude as to O*. For if these centres are P, Q, R, then the orthogonal PQ, passing through P cuts b, c, at equal angles, and, passing through Q cuts c, a, at equal angles. It therefore cuts a, b, at equal angles, and passes through R (§ 7).

The axis of similitude of a, b, c, as to O, like every circle cutting a, b, c, at equal angles, is orthogonal to the three bisectors, and is coaxal with the orthogonal circle and the ordinary axis of similitude of a, b, c.

11. The three bisectors are coaxal.

Also the bisector of b, $-c$, making supplementary angles with b, c, § 2 (f), makes equal angles with b, c, and is orthogonal to the bisector of b, c.

12. Let a circle x which cuts two circles a, b, at equal angles cut a at P, Q. Since the circle of inversion which inverts a into b inverts x into itself, the points P′, Q′ in which x cuts b are inverse to P, Q. Hence the right lines PP′, QQ′ go through the centre of

similitude of a, b. It follows that any circle through P and P′, being coaxal with the right line PP′ and the circle x, is orthogonal to the circle of inversion and makes equal angles with a, b. Hence orthogonals to O through P, P′ and Q, Q′ pass through the centre of similitude of a, b as to O.

13. Let a circle Σ be orthogonal to two circles A and B, whose radii are a and b, cutting A at P, Q and B at P′, Q′. Since Σ makes equal angles both with a, b and with a, $-b$, it follows from § 12 that the orthogonals PP′, QQ′ intersect at one of the centres of similitude as to O of A, B and the orthogonals PQ′, P′Q at the other.

On this simple theorem the following proofs of two fundamental theorems are based.

14. *The Pole and Polar Theorem.*

If through a fixed point E in the plane of a given circle C any two orthogonals be drawn cutting the circle C in P, Q and P′, Q′ respectively, then the intersections of the orthogonals PP′, QQ′ and those of PQ′, P′Q lie on a fixed orthogonal.

Proof. Draw the circles X,Y orthogonal to C; X through P and Q; Y through P′ and Q′.

Then, by § 13, the orthogonals PP′, QQ′ intersect at one of the centres of similitude as to O of X, Y and the orthogonals PQ′, P′Q at the other. Hence by § 8 the orthogonal through the intersections is the line of centres as to O of X and Y.

Now the circle D orthogonal to C and to two of the orthogonals through E (*i.e.*, the circle orthogonal to C, and with centre as to O at E) is orthogonal to all the orthogonals through E; being orthogonal to PQ and C, it is orthogonal to X, which is coaxal with them.

Since X, and similarly Y, are orthogonal to C and D, they belong to a fixed coaxal system, and, § 6, their line of centres as to O is a fixed orthogonal, which proves the theorem.

15. The proof only requires that X and Y should be orthogonal to a fixed circle besides C. This will be ensured if the circles PQ are orthogonal to a fixed circle H, which is also orthogonal to C; for then X will be orthogonal to H.

In particular, the proof will hold if the circles PQ are coaxal, even if O is not one of their orthogonal circles.

16. The fixed orthogonal of the theorem of § 14 may be called the O - polar of E with respect to C.

All the well known graphical developments follow from the theorem in the ordinary way.

The polar goes through the points of contact of the orthogonals through E touching C. Hence, if an orthogonal cut a circle a in X and X′, the O - pole with respect to a of the orthogonal XX′ is the centre as to O of the orthogonal to a through X, X′.

The case when C is the base O itself is important in certain applications of the theory, *e.g.*, reciprocation. The O - polar of E in that case is the orthogonal whose centre as to O is at E; conversely the O - pole of an orthogonal is its centre as to O. Two orthogonals are therefore conjugate, *i.e.*, the O - pole of either will lie on the other, if they are orthogonal to each other.

17. *Lemma for Pascal's Theorem.*

Let 1, 2, 3, ..., n be points in a right line, not necessarily in order; on 12, 23, ..., $\overline{n-1n}$, $n1$ as diameters describe circles. Let (12), (23), etc., be their radii with signs chosen as follows:— (12) arbitrarily, the rest in turn, so that the circle (23) touches (12), (34) touches (23), ..., $(n1)$ touches $(\overline{n-1\ n})$.

Will $(n1)$ *touch* (12)?

By § 2 (d), we have $21 : 23 = (12) : (23)$
where 21, 23 are signed segments on the right line.

$$\begin{aligned}
\text{Thus}\qquad & 12 : 23 = -(12) : (23) \\
\text{Similarly}\qquad & 23 : 34 = -(23) : (34) \\
& 34 : 45 = -(34) : (45) \\
& \quad\cdots\cdots\cdots \\
& \overline{n-1n} : n1 = -(\overline{n-1n}) : (n1).
\end{aligned}$$

Hence, compounding, we have

$$12 : n1 = (-1)^{n-1}(12) : (n1).$$

For contact of (12), $(n1)$, we need

$$12 : n1 = -(12) : (n1).$$

Therefore, the end members (12), ($n1$) of the chain will touch *provided n is even.*

By inversion, we find the same result for n points on a circle, the circles 12, 23, etc., being now orthogonals to that circle.

18. *Pascal's Theorem.*

Let 1, 2, 3, 4, 5, 6 be any six points on a circle Σ; then will the intersections of the pairs of orthogonals to O, 12, 45; 23, 56; and 34, 61 lie on an orthogonal.

Draw the orthogonals to Σ through 14, 25, 36.

By § 13, if the orthogonals 12, 45 intersect at P; 23, 56 at Q; 34, 61 at R, and 31, 64 at R′; then P is one of the O - centres of similitude of 14, 25; and so on.

Hence the orthogonal PQ goes through either R or R′; we can determine which it is, by the above lemma.

For that lemma shows that we can choose signs so that each of the circles (12), (23), (34), (45), (56), (61) touches its two neighbours; also that we can take (14) to touch (12) and (34), and therefore to touch (61) and (45) also, (§ 9); similarly with (25) and (36).

Then, since (12), (45) both touch (14) and (25), the point pair P where the orthogonals through 12, 45 intersect is the centre of similitude as to O of (14), (25), (§ 12). Hence PQR is an orthogonal, viz., the axis of similitude as to O of (14), (25), (36).

19. It may happen that PQR′ is also an orthogonal; that is to say, that PQRR′ lie on an orthogonal, This orthogonal, passing through both centres of similitude as to O of 14, 36, viz., R and R′, is their line of centres as to O, and cuts them at right angles; hence PQRR′ cuts all the circles 14, 25, 36 at right angles, and passes through their O - centres. The circles 14, 25, 36 are therefore coaxal.

20. *Brianchon's Theorem* is derived in the usual way by combination of Pascal's Theorem with the Pole and Polar Theorem.

21. *The Apollonian problem*: to describe a circle to touch three given circles a, b, c.

Analysis. Suppose that ρ is such a circle touching a, b, c at X, Y, Z. Since ρ makes equal angles with a, b, c, it is orthogonal to each of the bisecting circles of a, b, c, (§ 10). If then Σ is the

circle orthogonal to a, b, c, and therefore orthogonal to the bisectors, the centre of ρ as to Σ is at the common point pair I,I′ of the bisector.

The circle II′X is then orthogonal to ρ, and so to a. Hence X must be one of the points where the circle coaxal with the bisecting circles, and orthogonal to a, cuts a.

Synthesis. Coaxal with the bisecting circles draw the circle orthogonal to a, to meet a at X and X′. The circle through X orthogonal to the bisecting circles will touch a, b, c. For this circle, being orthogonal to the bisecting circles, is orthogonal to the circle through XX′ coaxal with them, and the sign of its radius may therefore be chosen so that it touches a at X; and it makes equal angles with a, b, c.

22. The obvious point of interest in the construction is its perfect analogy with the corresponding construction for a plane rectilineal, or a spherical triangle. To bring out the analogy even more explicitly, let ABC be one of the circular triangles formed by a, b, c; the construction amounts to this:—bisect two of the angles B, C by circles orthogonal to Σ; from I, their common point, draw IX orthogonal to BC and Σ; then the circle through X, with centre as to Σ at I, touches BC, CA, and AB.

23. The *Gergonne construction* is easily deduced, and generalised,

For X, X′ are the common points of two circles orthogonal to Σ, viz., a and IXX′; hence the right line XX′ passes through the centre of Σ. Also the pole of the line XX′ with respect to a, being the centre of the circle through XX′ orthogonal to a, that is of the circle IXX′, which is coaxal with the bisecting circles, lies on the line of centres of these, that is, on the axis of similitude. Thus XX′ is the right line through the radical centre and the pole of the axis of similitude with respect to a.

In the very same way, with orthogonals to O, the orthogonal through X, X′ passes through the centre of Σ as to O. Also the O - pole of the orthogonal XX′ with respect to a, being (§ 16) the O - centre of the circle through XX′ orthogonal to a, *i.e.*, of IXX′, lies on the axis of similitude of a, b, c as to O. Hence X, X′ are given as the intersections of a by the orthogonal through the radical centre as to O and the O - pole with respect to a of the O - axis of similitude.

24. *The Pascal lines of three pairs of points on a circle.*

In the figure of §18 suppose that the radii (14), (25), (36), or a, b, c, say, are assigned to begin with. Suppose also that the points 1, 4 are only given *as a pair*, viz., as the points of intersection of a and Σ; and similarly with 2, 5 and 3, 6. We can affix the marks 1, 4 arbitrarily; then so affix the marks 2, 5 that it is 12, not 15, which cuts (14), (25) at equal angles—by § 13 one of them does so—; then the marks 3, 6, so that it is 23, not 26, which cuts (25), (36) at equal angles. Hence, (14), (25), (36), or a, b, c being given, we can so take the points 1, 2, 3, 4, 5, 6, that each of the nine orthogonals to Σ, a, b, c, (12), ... (61) touches the four which it meets on Σ. And we see easily that *change of the sign* of a is simply equivalent to interchange of the marks 1 and 4 at the points of intersection of a and Σ, a suggestive result from the graphical point of view.

25. *The Gergonne construction in terms of co-orthogonals with the given circles.*

The construction stated in the last sentence of § 23 fails when the base O coincides with Σ, for we have not defined the centre of Σ as to Σ, nor the Σ - pole of one orthogonal with respect to another. But the original construction of § 21 may for this case be stated:—the orthogonal IXX′ joins the Σ - centres (or Σ - poles) of the Σ - axis of similitude and the orthogonal a. The whole construction may therefore be thus stated in terms of orthogonals to Σ:—

Let the given circles cut their orthogonal circle Σ in 14, 25, 36. Join the Σ - pole of the Pascal Σ - line of the Σ - hexagon 123456 by a Σ - line to the Σ - pole of 14. This cuts 14 in the points of contact of a pair of touching circles. The other three pairs of touching circles are found by interchanging 1, 4 or 2, 5 or 3, 6 in this construction.

An identical construction holds for the conics having double contact with a given conic Σ and touching three given lines which cut Σ in 14, 25, 36, Σ - lines being in this case simply right lines.

On Factors of Numbers of the Form $\{x^{(2n+1)k} \pm 1\} \div \{x^k \pm 1\}$.

By K. J. SANJANA, M.A.

(*Read* 12*th June* 1908).

1. In this paper the factorization of arithmetical numbers of the form $\{x^{2nk+k} \pm 1\} \div \{x^k \pm 1\}$, where x is a rational number such that kx is a perfect square, is investigated by means of a trigonometrical transformation. The number k will be taken to be prime for the present.

When $k \neq 2$, it can be at once shown that $(x^k \mp 1) \div (x \mp 1)$ may be a difference of two square numbers according as k is of form $4p \pm 1$. For let $(x^k - 1) \div (x - 1)$ or $x^{k-1} + x^{k-2} + x^{k-3} .... + 1$

$$\equiv \{x^{\frac{1}{2}(k-1)} + a_1 x^{\frac{1}{2}(k-3)} + a_2 x^{\frac{1}{2}(k-5)} \ldots + a_1 x + 1\}^2$$
$$- kx\{x^{\frac{1}{2}(k-3)} + b_1 x^{\frac{1}{2}(k-5)} .... + b_1 x + 1\}^2 ;$$

then $2a_1 - k = 1$ and $2a_2 + a_1^2 - 2b_1 k = 1$, whence $\frac{1}{4}(k+1)^2 = 1 - 2a_2 + 2b_1 k$, so that $\frac{1}{2}(k+1)$ is odd. Let it be $2p+1$, then $k = 4p+1$. Similarly we can show $k = 4p - 1$ or $4p + 3$ in the other case.

2. When k is 2 the following proposition holds good: the number $x^{4n+2} + 1$ has four rational factors, $2x$ being a perfect square.*

Let $2x = y^2$. We have $x^2 + 1 = x^2 + 2x + 1 - y^2 = (x + y + 1)(x - y + 1)$.
It is easily seen that

$$z^4 + 1 = (z^2 - 2z\cos\frac{\pi}{4} + 1)(z^2 - 2z\cos\frac{3\pi}{4} + 1), \text{ and}$$

$$z^{8n+4} + 1 = (z^{4n+2} - 2z^{2n+1}\cos\frac{\pi}{4} + 1)(z^{4n+2} - 2z^{2n+1}\cos\frac{3\pi}{4} + 1).$$

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* See the author's question in the *Educational Times* for June 1898. Numbers of the form $x^{4n+2} + 1$ have been called *Bin-Aurifeuillians* by Lt.-Col. Allan Cunningham, R.E., who has dealt with them in a paper "On Aurifeuillians" in the *Proceedings of the Lond. Math. Soc.*, Vol. XXIX. (March 1898). My acknowledgments are due to Lt.-Col. Cunningham for his kindness in allowing me to draw on this paper and on his solutions in the Reprints from the *Educ. Tim.* for most of my illustrative examples.

Divide the latter equation by the former, and put $z^2 = x$; we thus get (1)

$$\frac{x^{4n+2}+1}{x^2+1} = \frac{\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{4} + 1\right\}\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{3\pi}{4} + 1\right\}}{\left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{4} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{3\pi}{4} + 1\right)}.$$

Now $\cos(2n+1)\frac{\pi}{4} = \cos\frac{\pi}{4}$ or $\cos\frac{3\pi}{4}$, according as n is of forms $4p$, $4p+3$ or $4p+1$, $4p+2$; and in these cases

$$\cos(2n+1)\frac{3\pi}{4} = \cos\frac{3\pi}{4} \text{ or } \cos\frac{\pi}{4}.$$

Therefore the right hand expression has, for every form of n, the following value

$$\frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{4} + 1}{x - 2x^{\frac{1}{2}}\cos\frac{\pi}{4} + 1} \times \frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{3\pi}{4} + 1}{x - 2x^{\frac{1}{2}}\cos\frac{3\pi}{4} + 1} \quad \ldots(\alpha).$$

But it can be readily demonstrated that

$$x^{2n-2} + \frac{\sin 2\theta}{\sin\theta}x^{2n-3} + \frac{\sin 3\theta}{\sin\theta}x^{2n-4} \ldots + \frac{\sin n\theta}{\sin\theta}x^{n-1}$$
$$+ \frac{\sin(n-1)\theta}{\sin\theta}x^{n-2} \ldots + \frac{\sin 2\theta}{\sin\theta}x + 1 = \frac{x^{2n} - 2x^n\cos n\theta + 1}{x^2 - 2x\cos\theta + 1}.$$

Making requisite changes we obtain for (α)

$$\left\{x^{2n} + \frac{\sin\frac{2\pi}{4}}{\sin\frac{\pi}{4}}x^{2n-\frac{1}{2}} + \frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}}x^{2n-1} + \frac{\sin\frac{4\pi}{4}}{\sin\frac{\pi}{4}}x^{2n-\frac{3}{2}} \ldots\ldots \frac{\sin\frac{3\pi}{3}}{\sin\frac{\pi}{4}}x + \frac{\sin\frac{2\pi}{4}}{\sin\frac{\pi}{4}}x^{\frac{1}{2}} + 1\right\}$$

$$\times\left\{x^{2n} + \frac{\sin\frac{6\pi}{4}}{\sin\frac{3\pi}{4}}x^{2n-\frac{1}{2}} + \frac{\sin\frac{9\pi}{4}}{\sin\frac{\pi}{4}}x^{2n-1} \ldots\ldots + \frac{\sin\frac{6\pi}{4}}{\sin\frac{3\pi}{4}}x^{\frac{1}{2}} + 1\right\}.$$

It is seen that all terms containing $x^{\frac{1}{2}}$ have $2^{\frac{1}{2}}$ in their coefficients, and that each of these terms has contrary signs in the two factors, and that the coefficients of the integral powers are all rational. Thus (a) is the product of two expressions of the form $P + \sqrt{2x}Q$ and $P - \sqrt{2x}Q$ or $P + Qy$ and $P - Qy$, where P and Q are rational integral functions of x of degree $2n$ and $2n - 1$ respectively. We see therefore that the left side of equation (1) has two rational factors; and as $x^2 + 1$ has been shown to have two factors, it follows that $x^{4n+2} + 1$ has four rational factors.

3. These factors may be evaluated for given arithmetical values of x and n. It will be found that

$$(x^6+1)/(x^2+1) = (x^2+x+1)^2 - y^2(x+1)^2;$$
$$(x^{10}+1)/(x^2+1) = (x^4+x^3-x^2+x+1)^2 - y^2(x^3+1)^2;$$
$$(x^{14}+1)/(x^2+1) =$$
$$(x^6+x^5-x^4-x^3-x^2+x+1)^2 - y^2(x^5-x^3-x^2+1)^2;$$
$$(x^{18}+1)/(x^2+1) =$$
$$(x^8+x^7-x^6-x^5+x^4-x^3-x^2+x+1)^2 - y^2(x^7-x^5-x^2+1)^2;$$
$$(x^{22}+1)/(x^2+1) = (x^{10}+x^9-x^8-x^7+x^6+x^5+x^4-x^3-x^2+x+1)^2$$
$$- y^2(x^9-x^7+x^5+x^4-x^2+1)^2;$$

and further similar identities can be easily obtained.

4. *Examples.*—(a) $242^{10} + 1$.

Here $x = 242,\ y = \sqrt{2x} = 22;$

hence $(242^{10}+1) \div 58\ 565$

$$= (242^4 + 242^3 - 242^2 + 242 + 1)^2 - 22^2(242^3+1)^2$$
$$= (3\ 443\ 856\ 263)^2 - (311\ 794\ 758)^2:$$

so that $N \equiv 242^{10} + 1$

$$= 5 \times 13 \times 17 \times 53 \times 5 \times 626\ 412\ 301 \times 3\ 755\ 651\ 021.$$

It may be shown* that $626\ 412\ 301 = 4\ 561 \times 137\ 341$, and $3\ 755\ 651\ 021 = 881 \times 4\ 262\ 941$. The last number is prime; so that the prime factors of N are

$$5^2,\ 13,\ 17,\ 53,\ 881,\ 4\ 561,\ 137\ 341,\ 4\ 262\ 941.$$

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* See *Reprints E. T.*, Vol. LXX. (Lt.-Col. Cunningham).

(*b*) $50^{14}+1$. Here $50^2+1=41\times 61$; also $x^6+x^5-x^4-x^3-x^2+x+1$ $=15\,931\,122\,551$, and $y(x^5-x^3-x^2+1)=3\,123\,725\,010$; so that the other two factors are 19 054 847 561 and 12 807 397 541. It will be found that 29 is a divisor of the given number; hence $50^{14}+1=29\times 41\times 61\times 657\,063\,709\times 12\,807\,397\,541$. There is no other small factor less than 151.

(*c*) $$N\equiv 9^{14}+8^{14}=8^{14}\{(\tfrac{9}{8})^{14}+1\}.$$

Let $x=\frac{9}{8}$, then $y=\sqrt{2x}=\frac{3}{2}$. Therefore

$$x^{14}+1=(x+y+1)(x-y+1)\{(x^6+x^5\ldots+1)^2-y^2(x^5-x^3\ldots 1)^2\}$$

$$=\frac{29}{8}\cdot\frac{5}{8}\cdot\frac{480\ 229}{8^6}\cdot\frac{391\ 693}{8^6}.$$

Multiplying out by 8^{14} we obtain

$$9^{14}+8^{14}=5\times 29\times 281\times 1\ 709\times 391\ 693.$$

The last number is prime.

It is evident that numbers of the form $x^{4n+2}+y^{4n+2}$, where $2xy$ is a perfect square, can be factorized by the method here adopted.

(*d*) $N\equiv 18^{18}+1$. Here $18^2+1=5^2\times 13$;

$$18^8+18^7-18^6-18^5+18^4-18^3-18^2+18+1=11\ 596\ 377\ 655,$$

and $$6(18^7-18^5-18^2+1)=3\ 661\ 980\ 846;$$

so that the two large factors are

$$(\alpha)\quad 15\ 258\ 501 \text{ and } (\beta)\quad 7\ 934\ 396\ 809.$$

Now N contains $18^6+1=34\,012\,225$; dividing by 325, we see that 104 653 is a factor. The prime factors of this are 229 and 457; and it will be found that

$$(\alpha)\div 457=33\ 388\ 093,$$
$$(\beta)\div 229=34\ 648\ 021.$$

It is also evident that 37 is a divisor of N; and the last number written down is twice divisible by 37. Thus we finally get

$$N=5^2\times 13\times 37^2\times 229\times 457\times 25\ 309\times 33\ 388\ 093.$$

The large factor has been shown* to be prime; so that the above resolution is ultimate.

* By Mr C. E. Bickmore (Lt.-Col. Cunningham's paper, *Proc. Lond. Math. Soc.* XXIX.).

(e) $$N \equiv 200^{18}+1=(200^2+1)(P^2-y^2Q^2).$$

It is seen that $200^2+1=13 \,.\, 17 \,.\, 181$;

that $P=2\ 572\ 735\ 681\ 591\ 960\ 201,$

and $yQ=\ \ 255\ 993\ 599\ 999\ 200\ 020.$

The two large factors are therefore

$$F_1 \equiv 2\ 828\ 729\ 281\ 591\ 160\ 221$$

and $$F_2 \equiv 2\ 316\ 742\ 081\ 592\ 760\ 181.$$

Now 200^6+1 divides N, and has the factors 44 221 and 36 181 ($\equiv 97 \times 373$) besides 200^2+1. It is thus found that

$$F_1=44\ 221 \times 63\ 968\ 007\ 996\ 001,$$

and $$F_2=97 \times 373 \times 64\ 032\ 008\ 004\ 001.$$

Writing the given number in the form

$$(40\ 000)^9+1 \equiv (37m+3)^9+1 \equiv (73m-4)^9+1,$$

we get 37 and 73 as further divisors. These will be found to divide into the large factor of F_1: so that we finally get

$$200^{18}+1=13 \times 17 \times 181 \times 44\ 221 \times 97 \times 373 \times 37 \times 73$$
$$\times 23\ 683\ 083\ 301 \times 64\ 032\ 008\ 004\ 001.$$

The large numbers have not been examined for factors.

(*f*) $32^{22}+1$. The factor $32^2+1=5^2 \,.\, 41$; the other two factors will be found to be (§ 3)

$$F_1 \equiv 1\ 441\ 151\ 891\ 495\ 977,$$

and $$F_2 \equiv \ \ 878\ 751\ 140\ 256\ 793.$$

It will be found* that 397 divides F_1 and 2 113 divides F_2. Thus we have

$$32^{22}+1=5^2 \,.\, 41 \,.\, 397 \,.\, 2\ 113 \,.\, 3\ 630\ 105\ 520\ 141 \,.\, 415\ 878\ 438\ 361.$$

The large factors have not been examined.

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* See *Proc. Math. Soc. Lond.* XXIX. (Lt.-Col. Cunningham).

(*g*) $N \equiv 8^{30} + 1$. The factor $8^2 + 1 = 5 \cdot 13$; the other two factors are $$(x^2+1)(x^2-1)^2(x^4+1)^2 + x(x^{12} - x^{10} + x^8 - x^6 + x^4 - x^2 + 1)$$ $\pm y(x^2-1)(x^4+1)(x^7-1)$, where $x = 8$, $y = 4$. I find these to be

$$F_1 = 7\ 036\ 872\ 740\ 045$$

and $$F_2 = 2\ 706\ 490\ 805\ 957.$$

Writing the number in the form $512^{10} + 1$, one factor is seen to be

$$512^2 + 1 = 5 \cdot 13 \cdot 37 \cdot 109.$$

Thus $\quad N = 5 \cdot 13 \cdot 37 \cdot 73\,148\,400\,161 \cdot 5 \cdot 109 \cdot 12\,911\,693\,101.$

As $8^{10} + 1$ is a factor of N, other divisors will be found to be 1321, 41, 61; and it may be shown that 181 is also a factor. Thus finally

$$N = 5^2 \cdot 13 \cdot 37 \cdot 109 \cdot 41 \cdot 61 \cdot 1\,321 \cdot 181 \cdot 54\,001 \cdot 29\,247\,661.$$

I have not examined the last number.

5. When k is a prime greater than 2, the following result holds good: the number $\{x^{(2n+1)k} \pm 1\} \div \{x^k \pm 1\}$ has* three rational factors, kx being a perfect square and the upper or lower sign being taken according as k is of form $4p - 1$ or $4p + 1$. Before considering the general theorem, I shall take up the cases when k is 3 and 5.

Let $3x = y^2$;

then $\quad (x^3 + 1)/(x + 1) = x^2 + 2x + 1 - y^2 = (x + y + 1)(x - y + 1).$

Also $\quad z^6 + 1 = \left(z^2 - 2z\cos\frac{\pi}{6} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{6} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{6} + 1\right);$

changing z to $x^{\frac{1}{2}}$ and transposing the middle factor,

$$\frac{x^3+1}{x+1} = \left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{6} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{5\pi}{6} + 1\right).$$

Similarly, putting $z = x^{\frac{1}{2}(2n+1)}$,

$$\frac{x^{6n+3}+1}{x^{2n+1}+1} = \left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{6} + 1\right\}\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{5\pi}{6} + 1\right\}.$$

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* Except when n has the value $kp + \frac{1}{2}(p-1)$, in which case the general process fails.

Hence (1) $$\frac{x^{6n+3}+1}{x^{2n+1}+1} \div \frac{x^3+1}{x+1}$$

$$= \frac{\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{6} + 1\right\}\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{5\pi}{6} + 1\right\}}{\left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{6} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{5\pi}{6} + 1\right)}.$$

Now $\cos(2n+1)\frac{\pi}{6} = \cos\frac{\pi}{6}$ or $\cos\frac{5\pi}{6}$, when n is of form $6p$, $6p+5$ or $6p+2$, $6p+3$: and $\cos(2n+1)\frac{5\pi}{6} = \cos\frac{5\pi}{6}$ or $\cos\frac{\pi}{6}$ in the same cases respectively. Therefore the right hand expression has, for these forms of n, the following value

$$\frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{6} + 1}{x - 2x^{\frac{1}{2}}\cos\frac{\pi}{6} + 1} \times \frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{5\pi}{6} + 1}{x - 2x^{\frac{1}{2}}\cos\frac{5\pi}{6} + 1}.$$

Hence as in § 2, the left side of equation (1)

$$= \left\{x^{2n} + \frac{\sin\frac{2\pi}{6}}{\sin\frac{\pi}{6}}x^{2n-\frac{1}{2}} + \frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}x^{2n-1} \ldots\ldots + \frac{\sin\frac{2\pi}{6}}{\sin\frac{\pi}{6}}x^{\frac{1}{2}} + 1\right\}$$

$$\times \left\{x^{2n} + \frac{\sin\frac{10\pi}{6}}{\sin\frac{5\pi}{6}}x^{2n-\frac{1}{2}} + \frac{\sin\frac{15\pi}{6}}{\sin\frac{5\pi}{6}}x^{2n-1} \ldots\ldots + \frac{\sin\frac{10\pi}{6}}{\sin\frac{5\pi}{6}}x^{\frac{1}{2}} + 1\right\}.$$

It is seen that the coefficients of x^{2n}, x^{2n-1}, x^{2n-2} ... are absolutely equal and rational in the two brackets; and that the coefficients of the fractional powers are equal, but opposite in sign and involve $3^{\frac{1}{2}}$ throughout. Thus the expression above is the product of two factors of the form $P + \sqrt{3x}Q$ and $P - \sqrt{3x}Q$ or $P + yQ$ and $P - yQ$, where P and Q are rational integral functions of x of degree $2n$ and $2n-1$ respectively.

Again

$$\frac{x^{6n+3}+1}{x^3+1}=\left\{\frac{x^{6n+3}+1}{x^{2n+1}+1}\div\frac{x^3+1}{x+1}\right\}\times\frac{x^{2n+1}+1}{x+1}$$

$$=(\mathrm{P}+y\mathrm{Q})(\mathrm{P}-y\mathrm{Q})(x^{2n}-x^{2n-1}+x^{2n-2}\ldots-x+1),$$

so that it is the product of three rational factors each of degree $2n$; and as x^3+1 has been shown to have three factors, it follows that* $x^{6n+3}+1$ has six rational factors, when n has one of the forms given above.

6. Putting $n=2$, 3, 5, and evaluating the coefficients above obtained, we get the following results:

$$(x^{15}+1)/(x^3+1)=(x^4-x^3+x^2-x+1)\times$$
$$\{(x^4+2x^3+x^2+2x+1)^2-y^2(x^3+x^2+x+1)^2\};$$
$$(x^{21}+1)/(x^3+1)=(x^6-x^5+x^4-x^3+x^2-x+1)\times$$
$$\{(x^6+2x^5+x^4-x^3+x^2+2x+1)^2-y^2(x^5+x^4+x+1)^2\};$$
$$(x^{33}+1)/(x^3+1)=(x^{10}-x^9+x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1)\times$$
$$\{(x^{10}+2x^9+x^8-x^7-2x^6-x^5-2x^4-x^3+x^2+2x+1)^2$$
$$-y^2(x^9+x^8-x^6-x^5-x^4-x^3+x+1)^2\}.$$

Other similar identities can be obtained without difficulty whenever $6n+3$ does not contain a power of 3 higher than the first.

7. *Examples*:

(*a*) $48^{15}+1$. Here $y=12$; $48^3+1=49\,.\,37\,.\,61$.

The other factors are $48^4-48^3+48^2-48+1$,

and $48^4+2\,.\,48^3+48^2+2\,.\,48+1\pm12(48^3+48^2+48+1)$.

The given number will thus be found to be

$7^2\,.\,37\,.\,61\,.\,31\,.\,134\,731\,.\,5\,200\,081\,.\,6\,887\,341$.

The last two numbers have not been examined.

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* See *Educ. Times* for August 1902. Numbers of this form have been called *Trin-Aurifeuillians* by Lt-Col. Cunningham in his paper "On Aurifeuillians" mentioned above. As before I have derived much help from that paper in my examples.

(*b*) $N^* = 972^{15} + 1$. Here $x = 972$, $y = 54$;

$$x^3 + 1 = (x + 1)(x + y + 1)(x - y + 1) = 7 \,.\, 139 \,.\, 919 \,.\, 13 \,.\, 79\,;$$

$$x^4 - x^3 + x^2 - x + 1 = 891\,699\,420\,421 = 1\,291 \,.\, 690\,704\,431,$$

where the large number is prime. It will be found that

$$(P + yQ)(P - yQ) = 944\,095\,306\,951 \times 844\,813\,520\,011\,;$$

and it is easily shown that

31, 151, 181, 211, 541

are divisors of the given number. By actual division we obtain

$$P + yQ = 31 \,.\, 151 \,.\, 181 \,.\, 211 \,.\, 5\,281\,;$$

and $\qquad P - yQ = 541 \,.\, 1\,561\,577\,671,$

where the large number has been shown to be prime. Thus the complete factorization of N is

$$7 \,.\, 139 \,.\, 919 \,.\, 13 \,.\, 79 \,.\, 1291 \,.\, 31 \,.\, 151 \,.\, 181 \,.\, 211 \,.\, 541 \,.\, 5\,281$$
$$\times\, 690\,704\,431 \times 1\,561\,577\,671.$$

(*c*) $12^{21} + 1$. Here $12^3 + 1 = 13 \,.\, 7 \,.\, 19$;

$$12^6 - 12^5 + \ldots - 12 + 1 = 2\,756\,293\,;$$

and the remaining two factors are

$$3\,502\,825 \pm 1\,617\,486, \text{ i.e., } 5\,120\,311 \text{ and } 1\,885\,339.$$

The first of these is divisible by 7, and the quotient has the factors 43 and 17 011. Thus

$$12^{21} + 1 = 7^2 \,.\, 13 \,.\, 19 \,.\, 43 \,.\, 17\,011 \,.\, 1\,885\,339 \,.\, 2\,756\,293.$$

(*d*) $3^{111} + 1$. The factors of $3^3 + 1$ are† 1, 4, 7; and

$$3^{36} - 3^{35} \ldots - 3 + 1 = 112\,570\,976\,472\,749\,341.$$

The other two factors are found to be

$$(3^{36} + 2 \,.\, 3^{35} + 3^{34} - 3^{33} - 2 \,.\, 3^{32} \ldots - 3^{17} - 2 \,.\, 3^{16} - 3^{15} + 3^{14} + 2 \,.\, 3^{13} \ldots + 2 \,.\, 3 + 1)$$
$$\pm\, 3(3^{35} + 3^{34} - 3^{33} - 3^{31} \ldots - 3^{19} - 3^{16} - 3^{15} + 3^{13} \ldots + 3 + 1),$$

i.e., $\qquad 450\,283\,904\,728\,735\,897$

and $\qquad 64\,326\,272\,436\,179\,833\,;$

223, a divisor of the number, is contained in the first of these, the quotient being $\qquad 2\,019\,210\,335\,106\,439.$

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* See *Reprints E. T.*, Vol. LXX. (Lt.-Col. Cunningham).

† 1 is algebraically a factor, though it does not count numerically.

There is no other divisor smaller than 251.

8. When n is of form $6p+1$ or $6p+4$, *i.e.*, $3q+1$, the index $6n+3=9(2q+1)$, and is therefore a power of 3 or a multiple of a power of 3. In this case $\cos(2n+1)\frac{\pi}{6}$ and $\cos(2n+1)\frac{5\pi}{6}$ are equal to $\cos\frac{\pi}{2}$ and $\cos\frac{5\pi}{2}$, and hence vanish; so that the right side of equation (1) cannot be put into the form

$$\frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{6} + 1}{x - 2^{x\frac{1}{2}}\cos\frac{\pi}{6} + 1} \times \frac{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{5\pi}{6} + 1}{x - 2x^{\frac{1}{2}}\cos\frac{5\pi}{6} + 1},$$

and the trigonometrical quotients of § 5 cannot be obtained. We may, however, proceed algebraically thus.

Let $6n+3=9(2q+1)$; then $(x^{6n+3}+1)/(x^3+1)$

$$=\frac{x^{9(2q+1)}+1}{x^{3(2q+1)}+1}\cdot\frac{x^{3(2q+1)}+1}{x^3+1}=\{x^{6(2q+1)}-x^{3(2q+1)}+1\}\,.\,\mathrm{E}_1\,.$$

As shown above, E_1 is a product of three rational factors; and the bracketed expression, being $\{x^{3(2q+1)}+1\}^2-y^2(x^{2q+1})^2$, is a difference of two squares. Hence the number $x^{6n+3}+1$ has 8* rational factors, including the three of x^3+1. But if $2q+1$ is itself a multiple of 3, $\{x^{3(2q+1)}+1\}\div(x^3+1)$ has five factors, and thus the given expression has ten. In general, if $6n+3=3^l f$, where $f \neq 3m$, I find the number of factors of $x^{6n+3}+1$, given by this process, to be $2(l+2)$; but when $6n+3=3^l$, the number is only $2l+1$. But this number can be increased to at least $2l+4$ by various artifices.

Examples.—

(*a*) $(75^9+1)\div(75^3+1)=(75^3+1-15.75)\,(75^3+1+15.75)$

and $75^3+1=(75+1)\,(75+1-15)\,(75+1+15)$.

Thus $75^9+1=76\,.\,61\,.\,91\,.\,420\,751\,.\,423\,001$

$=2^2\,.\,19\,.\,61\,.\,91\,.\,127\,.\,3\,313\,.\,423\,001.$

The last number is prime.

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* Except when q is zero.

(*b*) $$\frac{48^{27}+1}{48^3+1}=\frac{48^{27}+1}{48^9+1}\cdot\frac{48^9+1}{48^3+1}$$

$=(48^9+1-12.48^4)(48^9+1+12.48^4)(48^3+1-12.48)(48^3+1+12.48)$.

As $48^3+1=49\,.\,37\,.\,61$, we get

$48^{27}+1=7^2\,.\,37\,.\,61\,.\,19\,.\,5\,851\,.\,110\,017\,.\,F_1\,.\,F_2$, where

$F_1=1\,352\,605\,524\,295\,681$, $F_2=1\,352\,605\,396\,893\,697$.

(*c*) $12^{45}+1$. Let $x=12$, $\sqrt{3x}=y=6$; then

$$x^{45}+1=\frac{x^{45}+1}{x^{15}+1}\cdot\frac{x^{15}+1}{x^3+1}\,(x^3+1)$$

$$=(x^{15}+1-yx^7)\,(x^{15}+1+yx^7)\,\{x^4-x^3+x^2-x+1\}\times$$

$\{(x^4+2x^3+x^2+2x+1)^2-y^2(x^3+x^2+x+1)^2\}(x+1)\{(x+1)^2-y^2\}$, by §6.

But $x^{45}+1=(x^3)^{15}+1$, and $3x^3=y^2x^2$; hence

$$x^{45}+1=(x^3+1)\,\{(x^3+1)^2-x^2y^2\}\,\{x^{12}-x^9+x^6-x^3+1\}\times$$
$$\{(x^{12}+2x^9+x^6+2x^3+1)^2-x^2y^2(x^9+x^6+x^3+1)^2\},$$

by the same formula. Now x^3+1 contains the last three factors obtained by the first process, and $x^{12}-x^9+x^6-x^3+1$ the preceding three. Hence the large factors $x^{15}+1\mp yx^7$ are divisible by $(x^3+1)\mp xy$; it will be found that

$$x^{45}+1=(x+1)\,(x+1+y)\,(x+1-y)\,(x^4-x^3+x^2-x+1)\times$$
$$(x^4+2x^3+x^2+2x+1-y\,\overline{x^3+x^2+x+1})\,(x^4+2x^3+x^2+2x+1+y\,\overline{x^3+x^2+x+1}).$$
$$\times\,(x^3+1+xy)\,(x^3+1-xy)\times$$
$$(x^{12}+2x^9+x^6+2x^3+1-xy\,\overline{x^9+x^6+x^3+1})\,(x^{12}+2x^9+x^6+2x^3+1+xy\,\overline{x^9+x^6+x^3+1}).$$

We thus get $12^{45}+1$

$=13\,.\,19\,.\,7\,.\,19\,.\,141\,.\,13\,051\,.\,35\,671\,.\,1\,801\,.\,1\,657\times F_1\,.\,F_2$;

and $13\,051=31\,.\,421$; therefore,

$12^{45}+1=13\,.\,19\,.\,7\,.\,31\,.\,421\,.\,19\,141\,.\,35\,671\,.\,1\,801\,.\,1\,657\times$
$9\,298\,142\,299\,081\,.\,8\,554\,703\,697\,721$.

The last two numbers have not been tested.

(*d*) $3^{99}+1\equiv N$. As $N=27^{33}+1$, the number has the following six factors (§ 6):

28, 19, 37, 198 537 877 376 983, 292 582 128 285 019,
150 244 883 667 451.

$$\text{As } N=\frac{3^{99}+1}{3^{33}+1}\,(3^{33}+1)=(3^{33}+1)(3^{33}-3\,.\,3^{16}+1)(3^{33}+3\,.\,3^{16}+1),$$

and $3^{33}+1$ has six factors (§ 6), we get $N=4\,.\,1\,.\,7\,44\,287\,.\,176\,419\,.\,25\,411\,.\,5\,559\,060\,437\,415\,361\,.\,5\,559\,060\,695\,695\,687$.

Again 176 419 and 25 411 are prime; 44 287 = 67 . 661; these are the factors of 198 537 877 376 983. Other divisors* of the given number are found to be 397, 199, 4 357 : and it is seen that

$$292\,582\,128\,285\,019 = 199 \,.\, 4\,357 \,.\, 337\,448\,233,$$

and
$$150\,244\,883\,667\,451 = 397 \,.\, 378\,450\,588\,583.$$

The large quotients* have been verified to be prime; so that N is completely factorised into

$$2^2 \,.\, 7 \,.\, 19 \,.\, 37 \,.\, 67 \,.\, 661 \,.\, 25\,411 \,.\, 176\,419 \,.\, 199 \,.\, 4\,357 \,.\, 397 \,.\, 337\,448\,233 \,.\, 378\,450\,588\,583.$$

9. When k is 5, it may be shown that the number $x^{10n+5} - 1$ has six rational factors when $5x$ is a square ($\equiv y^2$) and the index does not contain a power of 5 higher than the first. The method of proof is not simple, and the general result cannot be easily exhibited.

As before

$$\frac{z^{10}-1}{z^2-1} = \left(z^2 - 2z\cos\frac{2\pi}{10} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{10} + 1\right) \times \left(z^2 - 2z\cos\frac{6\pi}{10} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{10} + 1\right);$$

change z^2 to x and x^{2n+1} respectively and divide: we thus obtain

$$(1) \quad \frac{x^{10n+5}-1}{x^{2n+1}-1} \div \frac{x^5-1}{x-1} = \frac{\Pi\left(x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{5} + 1\right)}{\Pi\left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{5} + 1\right)},$$

where the product Π contains four factors involving the cosines of $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$. Also $\frac{x^5-1}{x-1}$

$$= \left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{5} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{2\pi}{5} + 1\right) \times \left(x - 2x^{\frac{1}{2}}\cos\frac{3\pi}{5} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{4\pi}{5} + 1\right)$$

$$= (x^2 - \sqrt{5}\, x^{\frac{3}{2}} + 3x - \sqrt{5}\, x^{\frac{1}{2}} + 1) \times (x^2 + \sqrt{5}\, x^{\frac{3}{2}} + 3x + \sqrt{5}\, x^{\frac{1}{2}} + 1$$

$$= (x^2 + 3x + 1)^2 - 5x(x+1)^2 = \{x^2 + 3x + 1 - y(x+1)\}\{x^2 + 3x + 1 + y(x+1)\}.$$

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* See *Proc. Lond. Math. Soc.*, XXIX. (Lt.-Col. Cunningham.)

It will be found that $\cos(2n+1)\frac{\pi}{5} = \cos\frac{\pi}{5}$ or $\cos\frac{3\pi}{5}$ according as n is of forms $5p$, $5p+4$ or $5p+1$, $5p+3$;

and $$\cos(2n+1)\frac{3\pi}{5} = \cos\frac{3\pi}{5} \quad\text{or}\quad \cos\frac{\pi}{5}$$

under the same circumstances. Similarly

$\cos(2n+1)\frac{2\pi}{5} = \cos\frac{2\pi}{5}$ or $\cos\frac{4\pi}{5}$ and $\cos(2n+1)\frac{4\pi}{5} = \cos\frac{4\pi}{5}$ or $\cos\frac{2\pi}{5}$

in the same cases respectively. Hence the right hand side of (1) has, for these forms of n, the following value

$$(2)\qquad \frac{\Pi\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{5} + 1\right\}}{\Pi\left\{x^2 - 2x^{\frac{1}{2}}\cos\frac{\pi}{5} + 1\right\}};$$

that is, it is the product of

$$f\left(\frac{\pi}{5}\right) = x^{2n} + \frac{\sin\frac{2\pi}{5}}{\sin\frac{\pi}{5}}x^{2n-\frac{1}{2}} + \frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}}x^{2n-1}\ldots + \frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}}\,x + \frac{\sin\frac{2\pi}{5}}{\sin\frac{\pi}{5}}\,x^{\frac{1}{2}} + 1,$$

and three similar series $f\left(\frac{2\pi}{5}\right), f\left(\frac{3\pi}{5}\right), f\left(\frac{4\pi}{5}\right)$.

The product of the series

$$f\left(\frac{\pi}{5}\right), f\left(\frac{2\pi}{5}\right)$$

is found to be

$$x^{4n} + \sqrt{5}\,.\,x^{4n-\frac{1}{2}} + 2\,.\,x^{4n-1} + 0 - 2\,.\,x^{4n-2} - \sqrt{5}\,.\,x^{4n-\frac{5}{2}}\ldots + 1\,;$$

this is of the form $\mathrm{P} + \sqrt{5x}\,\mathrm{Q}$, where P and Q are rational functions of x of degree $4n$ and $4n-1$ respectively. It will be seen that the products of

$$f\left(\frac{3\pi}{5}\right), f\left(\frac{4\pi}{5}\right)$$

is the complementary expression $\mathrm{P} - \sqrt{5x}\,\mathrm{Q}$. Now

$$\frac{x^{10n+5}-1}{x^3-1} = \left\{\frac{x^{10n+5}-1}{x^{2n+1}-1} \div \frac{x^5-1}{x-1}\right\} \times \frac{x^{2n+1}-1}{x-1}$$

$$= (\mathrm{P}+y\mathrm{Q})(\mathrm{P}-y\mathrm{Q})(x^{2n} + x^{2n-1} + \ \ldots\ + x + 1),$$

so that it is the product of three rational factors ; and as $x^5 - 1$ has been proved to have three factors, it follows that $x^{10n+5} - 1$ has six rational factors.

In the excepted case $n \equiv 5p + 2$, and the index is of form $5^2(2p + 1)$. Here $\cos(2n + 1)\frac{\pi}{5}$ is -1 ; and the right hand side of (1) cannot be put into the form (2). The trigonometrical divisions, therefore, cannot be performed ; and we shall have to proceed algebraically as in § 8.

10. In the case of $x^{15} - 1$, we have $n = 1$; hence, by the previous section,

$$f\left(\frac{\pi}{5}\right) \times f\left(\frac{2\pi}{5}\right)^{*} = \left(x^2 + \frac{\sin 2a}{\sin a}\, x^{\frac{3}{2}} + \frac{\sin 3a}{\sin a}\, x + \frac{\sin 2a}{\sin a}\, x^{\frac{1}{2}} + 1\right) \times$$

$$\left(x^2 + \frac{\sin 4a}{\sin 2a}\, x^{\frac{3}{2}} + \frac{\sin 6a}{\sin 2a}\, x + \frac{\sin 4a}{\sin 2a}\, x^{\frac{1}{2}} + 1\right),$$

where $a = \pi/5$. Multiplying out and simplifying we get

$$x^4 + \sqrt{5}x^{\frac{7}{2}} + 2x^3 + \sqrt{5}x^{\frac{5}{2}} + 3x^2 + \sqrt{5}x^{\frac{3}{2}} + 2x + \sqrt{5}x^{\frac{1}{2}} + 1,$$

that is, $\quad x^4 + 2x^3 + 3x^2 + 2x + 1 + y(x^3 + x^2 + x + 1)$;

and the product of $f\left(\frac{3\pi}{5}\right)$ and $f\left(\frac{4\pi}{5}\right)$ will be found to be the complementary expression. Thus we have $(x^{15} - 1)/(x^5 - 1) =$

$$(x^2 + x + 1)\{(x^4 + 2x^3 + 3x^2 + 2x + 1)^2 - y^2(x^3 + x^2 + x + 1)^2\}.$$

Following the same method for $n = 3$ and $n = 4$, I get

$$(x^{35} - 1)/(x^5 - 1) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \times$$
$$\{(x^{12} + 2x^{11} - 2x^{10} - x^9 + 5x^8 + x^7 - 3x^6 + x^5 + 5x^4 - x^3 - 2x^2 + 2x + 1)^2$$
$$- y^2(x^{11} - x^9 + x^8 + 2x^7 - x^6 - x^5 + 2x^4 + x^3 - x^2 + 1)^2\} ;$$
$$(x^{45} - 1)/(x^5 - 1) = (x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \times$$
$$\{(x^{16} + 2x^{15} - 2x^{14} - x^{13} - 4x^{11} + 2x^{10} + 3x^9 - x^8 + 3x^7 + 2x^6 - 4x^5 - x^3 - 2x^2$$
$$+ 2x + 1)^2 - y^2(x^{15} - x^{13} - x^{11} - x^{10} + 2x^9 + 2x^6 - x^5 - x^4 - x^2 + 1)^2\}.$$

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* It should be noticed that in the series f the co-efficients recur reciprocally after the middle term.

11. *Examples.*—(*a*) $50\,000^5 - 1 = 49\,999 \times$
$$\{50\,000^2 + 3 \times 50\,000 + 1 \pm 500\,(50\,001)\}$$
$$= 49\,999 \,.\, 2\,525\,150\,501 \,.\, 2\,475\,149\,501.$$
The first large number* = 151 . 541 . 30 911, and the second* $= 11^2 . 131 . 156\,151$.

(*b*) $N \equiv 320^{15} - 1$. The factors of $320^5 - 1$ are 319, 90 521, 116 201; *i.e.*, 11, 29, 131, 691, 116 201; and the number $320^2 + 320 + 1 = 139 . 739$. The remaining two factors are found to be
$$11\,866\,432\,681 \text{ and } 9\,236\,775\,001\,;$$
the first of these* = 31 . 1 951 . 196 201, and the second* = 61 . 661 . 229 081.
Thus the number is completely factorized.

(*c*) $20^{35} - 1$. It will be found that $20^5 - 1 = 19 . 11 . 61 . 251$; and $20^6 + 20^5 \ldots + 1 = 29 . 71 . 32\,719$.
The two large factors are
$$6\,527\,898\,023\,267\,251, \text{ and } 2\,441\,576\,160\,715\,231.$$
There is no other divisor less than 251.

(*d*) $N \equiv 45^{25} - 1 = \{(45^{25} - 1) \div (45^5 - 1)\}(45^5 - 1)$. The number $45^5 - 1 = 2^2 . 11 . 2\,851 . 1\,471$. Let $45^5 = x$; then $5 . 45^5 = 5^6 . 3^{10}$, so that $\sqrt{5x} = 5^3 . 3^5 = 30\,375$. Hence
$$(x^5 - 1)/(x - 1) = \{x^2 + 3x + 1 + y(x + 1)\}\{x^2 + 3x + 1 - y(x + 1)\}$$
$$= 34\,056\,234\,511\,427\,251 \,.\, 34\,045\,024\,427\,772\,751.$$
There is no other divisor less than 200.

(*e*) $$N \equiv 5^{75} - 1 = \frac{5^{75} - 1}{5^{15} - 1} \cdot \frac{5^{15} - 1}{5^5 - 1} \cdot (5^5 - 1)$$
$$= \{5^{30} + 3 . 5^{15} + 1 + 5^8(5^{15} + 1)\}\{5^{30} + 3 . 5^{15} + 1 - 5^8(5^{15} + 1)\}$$
$$\times (5^2 + 5 + 1)\{(5^4 + 2 . 5^3 + 3 . 5^2 + 2 . 5 + 1)^2 - 5^2(5^3 + 5^2 + 5 + 1)^2\}$$
$$\times (5 - 1)\{5^2 + 3 . 5 + 1 + 5(5 + 1)\}\{5^2 + 3 . 5 + 1 - 5(5 + 1)\}$$
$$= 2^2 . 11 . 71 . 31 . 181 . 1\,741 . F_1 . F_2 .$$
Also $N = \dfrac{5^{75} - 1}{5^{25} - 1} \cdot \dfrac{5^{25} - 1}{5^5 - 1} \cdot (5^5 - 1)$, putting $5^5 = x$ and $5^6 = y^2$,
$$= (5^5 - 1)\{(x^2 + 3x + 1)^2 - y^2(x + 1)^2\} \times$$
$$(x^2 + x + 1)\{(x^4 + 2x^3 + 3x^2 + 2x + 1)^2 - y^2(x^3 + x^2 + x + 1)^2\}$$
$$= 2^2 . 11 . 71 . 9\,384\,251 . 10\,165\,751 . 9\,768\,751 . G_1 . G_2 .$$
It will be found that $9\,768\,751 = 31 . 181 . 1\,741$, $F_1 = G_1 \times 9\,384\,251$, and $F_2 = G_2 \times 10\,165\,751$.

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* *Reprints E.T.*, Vol. LXX. (Lt.-Col. Cunningham).

Other small divisors of the number are seen to be 101, 151, 251. Thus $\quad N = 2^3 . 11 . 71 . 31 . 181 . 1\,741 . 9\,384\,251 . 101 . 251 . 401 . 151 . 606\,705\,812\,851 . 99\,244\,414\,459\,501.$ The large factors have not been tested.

12. It is now easy to see that the number $x^{(2n+1)k} \pm 1$ has six rational factors. In the first place we have

$$\frac{x^k - 1}{x - 1} = \left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{k} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{3\pi}{k} + 1\right)\ldots\left(x - 2x^{\frac{1}{2}}\cos\frac{k-1}{k}\pi + 1\right),$$

when $k = 4p + 1$; and

$$\frac{x^k + 1}{x + 1} = \left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{2k} + 1\right)\left(x - 2x^{\frac{1}{2}}\cos\frac{3\pi}{2k} + 1\right)\ldots\left(x - 2x^{\frac{1}{2}}\cos\frac{2k-1}{2k}\pi + 1\right)$$

when $k = 4p + 3$. In the second case the factor containing $\cos\frac{k\pi}{2k}$ is absent from the right side, being in fact the denominator of the left; thus the number of trigonometrical factors is $k - 1$ in both cases. It will be found that these can always be arranged in two groups each of $\frac{1}{2}(k-1)$ factors whose products are severally of forms $P + \sqrt{kx}Q$, $P - \sqrt{kx}Q$, where P and Q are rational functions of x of degree $\frac{1}{2}(k-1)$ and $\frac{1}{2}(k-3)$ respectively. Thus $(x^k \mp 1)/(x \mp 1)$ has two rational factors. In the next place, we put $x^{(2n+1)k} \pm 1$ in the form (A)

$$\left\{\frac{x^{(2n+1)k} \pm 1}{x^{2n+1} \pm 1} \div \frac{x^k \pm 1}{x \pm 1}\right\}\frac{x^{2n+1} \pm 1}{x \pm 1}(x^k \pm 1).$$

As before, we can prove

$$\frac{x^{(2n+1)k} + 1}{x^{2n+1} + 1} = \left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{2k} + 1\right\}\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{3\pi}{2k} + 1\right\}$$
$$\ldots\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{2k-1}{2k}\pi + 1\right\},$$

and a similar result for $\{x^{(2n+1)k} - 1\}/\{x^{2n+1} - 1\}$. Hence the expression in large brackets in (A) is

$$\frac{\Pi\left\{x^{2n+1} - 2x^{\frac{1}{2}(2n+1)}\cos\frac{\pi}{2k} + 1\right\}}{\Pi\left\{x - 2x^{\frac{1}{2}}\cos\frac{\pi}{2k} + 1\right\}} \quad \text{or} \quad \frac{\Pi\left\{x^{2n+1} - 2x^{\frac{1}{2}}\cos\frac{\pi}{k} + 1\right\}}{\Pi\left\{x - 2x^{\frac{1}{2}}\cos\frac{\pi}{k} + 1\right\}}$$

according as k is $4p+3$ or $4p+1$. The products Π contain each an even number of factors $k-1$; and as

$$\cos(2n+1)\frac{\pi}{2k},\ \cos(2n+1)\frac{3\pi}{2k},\ \ldots\ \cos(2n+1)\frac{2k-1}{2k}\pi$$

have the values

$$\cos\frac{\pi}{2k},\ \cos\frac{3\pi}{2k},\ \ldots\cos\frac{2k-1}{2k}\pi$$

in same order depending on the values of k*, as also

$$\cos(2n+1)\frac{\pi}{k},\ \cos(2n+1)\frac{2\pi}{k},\ \ldots\ \cos(2n+1)\frac{k-1}{k}\pi$$

have the values $\cos\frac{\pi}{k}$, $\cos\frac{2\pi}{k}$, $\ldots\cos\frac{k-1}{k}\pi$ in same order, it follows that the products take the form

$$\frac{\Pi\left\{x^{2n+1}-2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{2k}+1\right\}}{\Pi\left\{x-2x^{\frac{1}{2}}\cos\frac{\pi}{2k}+1\right\}}$$

or
$$\frac{\Pi\left\{x^{2n+1}-2x^{\frac{1}{2}(2n+1)}\cos(2n+1)\frac{\pi}{k}+1\right\}}{\Pi\left\{x-2x^{\frac{1}{2}}\cos\frac{\pi}{k}+1\right\}}.$$

Thus the expression is the product of $k-1$ series of the form

$$x^{2n}+\frac{\sin\frac{2\pi}{2k}}{\sin\frac{\pi}{2k}}x^{2n-\frac{1}{2}}+\frac{\sin\frac{3\pi}{2k}}{\sin\frac{\pi}{2k}}x^{2n-1}\ldots+\frac{\sin\frac{2\pi}{2k}}{\sin\frac{\pi}{2k}}x^{\frac{1}{2}}+1,$$

or
$$x^{2n}+\frac{\sin\frac{2\pi}{k}}{\sin\frac{\pi}{k}}x^{2n-\frac{1}{2}}+\frac{\sin\frac{3\pi}{k}}{\sin\frac{\pi}{k}}x^{2n-1}\ldots+\frac{\sin\frac{2\pi}{k}}{\sin\frac{\pi}{k}}x^{\frac{1}{2}}+1,$$

* The value of $\cos(2n+1)\frac{k\pi}{2k}$ is zero, and the factor corresponding to this function does not occur in the product.

in the two cases. Hence, always the expression referred to is the product of $k-1$ such trigonometrical series. It will be found* that these can always be arranged in two groups, each of $\frac{1}{2}(k-1)$ series, such that the products in the groups are of forms $P' + \sqrt{kx}Q'$, $P' - \sqrt{kx}Q'$, where P' and Q' are rational functions of x of degree $n(k-1)$ and $n(k-1)-1$ respectively. Thus

$$x^{(2n+1)k} \pm 1 = \{P' + \sqrt{kx}Q'\}\{P' - \sqrt{kx}Q'\}(x^{2n} \mp x^{2n-1} \dots + 1)(x^k \pm 1);$$

and as the last factor has been shown to have three rational factors, it follows that the given number has six such factors.

13. I conclude by giving a few formulæ for the values 7, 11, 13 of k.

The expression $x^{14}+1$ is the product of seven factors of the form $x^2 - 2x\cos\frac{k\pi}{14} + 1$; of these the central factor is x^2+1. Hence $(x^{14}+1)/(x^2+1)$ is the product of six such factors; and changing x^2 to x we get

$$(x^7+1)/(x+1) = \left(x - 2\sqrt{x}\cos\frac{\pi}{14} + 1\right)\left(x - 2\sqrt{x}\cos\frac{3\pi}{14} + 1\right)$$
$$\times\left(x - 2\sqrt{x}\cos\frac{5\pi}{14} + 1\right)\left(x - 2\sqrt{x}\cos\frac{9\pi}{14} + 1\right)\left(x - 2\sqrt{x}\cos\frac{11\pi}{14} + 1\right)$$
$$\left(x - 2\sqrt{x}\cos\frac{13\pi}{14} + 1\right)$$

It will be found that the factors containing $\frac{\pi}{14}, \frac{3\pi}{14}, \frac{9\pi}{14}$ give rise to a product of the form $P + \sqrt{7x}Q$; the others to the complementary expression. Hence

$$x^7 + 1 = (x+1)\{(x^3+3x^2+3x+1)^2 - 7x(x^2+x+1)^2\}.$$

Also, changing x^2 to x^3 we get

$$(x^{21}+1)/(x^3+1) = \Pi\left(x^3 - 2x^{\frac{3}{2}}\cos\frac{\pi}{14} + 1\right),$$

where there are six factors. Thus

$$\frac{x^{21}+1}{x^3+1} \div \frac{x^7+1}{x+1} = \frac{\Pi\left(x^3 - 2x^{\frac{3}{2}}\cos\frac{\pi}{14} + 1\right)}{\Pi\left(x - 2x^{\frac{1}{2}}\cos\frac{\pi}{14} + 1\right)}.$$

* I have no right proof to offer of this statement.

As $\cos 3 \cdot \frac{\pi}{14} = \cos\frac{3\pi}{14}$, $\cos 3 \cdot \frac{3\pi}{14} = \cos\frac{9\pi}{14}$, $\cos 3 \cdot \frac{9\pi}{14} = \cos\frac{\pi}{14}$,

it is seen that the three factors above involving

$$\frac{\pi}{14}, \frac{3\pi}{14}, \frac{9\pi}{14}$$

are divisible by the factors below involving

$$\frac{9\pi}{14}, \frac{\pi}{14}, \frac{3\pi}{14}$$

respectively: similar remarks apply to the remaining three factors. Hence the above quantity is the product of the following two groups of series

$$f\left(\frac{\pi}{14}\right), f\left(\frac{3\pi}{14}\right), f\left(\frac{9\pi}{14}\right) \text{ and } f\left(\frac{5\pi}{14}\right), f\left(\frac{11\pi}{14}\right), f\left(\frac{13\pi}{14}\right).$$

The former product will be found to be

$$x^6 + 4x^5 - x^4 - 7x^3 - x^2 + 4x + 1 + \sqrt{7x}(x^5 + x^4 - 2x^6 - 2x^2 + x + 1);$$

the latter to be the complementary expression. Hence finally

$$x^{21} + 1 = (x^7 + 1)(x^2 - x + 1) \times$$
$$\{(x^6 + 4x^5 - x^4 - 7x^3 - x^2 + 4x + 1)^2 - 7x(x^5 + x^4 - 2x^3 - 2x^2 + x + 1)^2\}.$$

Similarly changing x^3 to x^5 in the original identity, I find that

$$x^{35} + 1 = (x^7 + 1)(x^4 - x^3 + x^2 - x + 1) \times$$
$$\{(x^{12}+4x^{11}+6x^{10}+11x^9+15x^8+17x^7+19x^6+17x^5+15x^4+11x^3+6x^2+4x+1)^2$$
$$- 7x(x^{11} + 2x^{10} + 3x^9 + 5x^8 + 6x^7 + 7x^6 + 7x^5 + 6x^4 + 5x^3 + 3x^2 + 2x + 1)^2\}.$$

When $k = 11$, $(x^{11} + 1)/(x + 1) = \Pi\left(x^2 - 2x\cos\frac{k\pi}{22} + 1\right)$,

where Π includes 10 factors. The product of five of these involving the cosines of

$$\frac{\pi}{22}, \frac{5\pi}{22}, \frac{7\pi}{22}, \frac{9\pi}{22} \text{ and } \frac{19\pi}{22}$$

will be found to be

$$x^5 + 5x^4 - x^3 - x^2 + 5x + 1 + \sqrt{11x}(x^4 + x^3 - x^2 + x + 1);$$

that of the other five is the complementary surd. When $k=13$, we have to combine the six factors involving the cosines of

$$\frac{\pi}{13},\ \frac{2\pi}{13},\ \frac{3\pi}{13},\ \frac{6\pi}{13},\ \frac{8\pi}{13} \text{ and } \frac{9\pi}{13},$$

as also the six remaining ones. The following result is thus obtained

$$x^{13}-1=(x-1)\{(x^6+7x^5+15x^4+19x^3+15x^2+7x+1)^2 \\ -13x(x^5+3x^4+5x^3+5x^2+3x+1)^2\}.$$

The Intrinsic Properties of Curves

By Professor M. T. Naraniengar

(*Read 8th May, 1908*)

Any property of a curve which depends solely upon its form may be styled intrinsic. Thus the circle of curvature at a point, the relation between s and ψ, the envelope of the normals, the envelope of straight lines making a constant angle with the curve, etc., are all intrinsic properties. It is proposed to investigate a few of these properties in this paper.*

I.

(1) Let a straight line PQ make a constant angle λ with a given curve at any point P; then the envelope of PQ may be geometrically found as follows.

If PC, P′C (Fig. 17) be consecutive normals to the given curve, and PQ, P′Q consecutive positions of the straight line making a constant angle with the curve, then evidently TPCQP′ is a circle. Therefore $\angle$ CQT is right and ultimately $\angle$ CQP is a right angle. Hence the envelope touches the straight line at the foot of the perpendicular from C, the centre of curvature Also $PQ = \rho\sin\lambda$, where ρ is the radius of curvature at P.

(2) Again, if Q′ be a neighbouring point on the envelope,

$$QQ' = P'Q' - PQ + PP'\cos\lambda,$$

i.e., $$\delta s' = (\rho + \delta\rho)\sin\lambda - \rho\sin\lambda + \delta s\cos\lambda = \delta\rho\sin\lambda + \delta s\cos\lambda$$

the accented letters referring to the locus of Q.

Also $\delta\psi' = \delta\psi$, since $\psi' - \psi = \lambda$

$$\therefore \frac{\delta s'}{\delta\psi'} = \frac{\delta\rho}{\delta\psi}\sin\lambda + \frac{\delta s}{\delta\psi}\cos\lambda$$

$$\therefore \rho' \text{ or } \frac{ds'}{d\psi'} = \rho_1\sin\lambda + \rho\cos\lambda, \text{ where } \rho_1 = \frac{d\rho}{d\psi}$$

which is equal to the radius of curvature of the evolute.

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* [Full references to the literature of the problem dealt with in I. will be found in Loria, *Spezielle algebraische und transscendente ebene Kurven*. Leipzig, 1902, p. 626. Ed. *E.M.S.P.*]

(3) Let E be the centre of curvature of the evolute; then, if EN be perpendicular to QC, $QN = \rho\cos\lambda + \rho_1\sin\lambda$, so that N is the centre of curvature of the locus of Q. In other words, the centre of curvature at Q is the projection of the centre of curvature of the evolute on the normal at Q; and it is related to the evolute just as Q is related to the original curve.

(4) Further, since $\delta s' = \delta\rho\sin\lambda + \delta s\cos\lambda$, by integrating we have $s' = \rho\sin\lambda + s\cos\lambda +$ a constant. From this the intrinsic equation to the envelope of PQ may be expressed.

(5) From the relation $\rho' = \rho_1\sin\lambda + \rho\cos\lambda_1$ we can deduce that

$$\text{(i)} \qquad \frac{d\rho'}{d\psi} = \rho_2\sin\lambda + \rho_1\cos\lambda$$

$$\text{(ii)} \qquad \frac{d^2\rho'}{d\psi^2} = \rho_3\sin\lambda + \rho_2\cos\lambda$$

and so on; which means that the centre of curvature of the evolute of Q is the projection of the centre of curvature of the second evolute of the original curve, and so on.

(6) The locus of Q may be regarded as the involute of a curve similarly related to the evolute of the original. Also a system of parallel curves (P) will have a system of parallel curves, (Q), whose evolutes, viz., (C) and (N) are similarly related.

These considerations show that the locus of N may be styled an *oblique* (λ) evolute of the original curve P.

II.

(1) Let PQ be the diameter of curvature at P and CD that of the evolute, then the locus of Q will have QD for normal.

For if PQ, P′Q′ (Fig. 18) be consecutive positions of the diameter, then $PQ = 2\rho$ and $P'Q' = 2(\rho + \delta\rho)$. By projection we have in the limit

$$\tan\phi = \mathrm{L}\,\frac{QR}{Q'R} = \mathrm{L}\,\frac{\rho\delta\psi}{2\delta\rho} = \frac{\rho}{2\rho_1}$$

Hence $\phi = \angle\,QDC$, that is DQ is the normal at Q.

(2) If QQ' be denoted by $\delta s'$,

$$(\delta s')^2 = QR^2 + Q'R^2$$

$$\therefore \left(\frac{\delta s'}{\delta\psi}\right)^2 = \left(\frac{\delta s}{\delta\psi}\right)^2 + 4\left(\frac{\delta\rho}{\delta\psi}\right)^2 \text{ and } \therefore \left(\frac{ds'}{d\psi}\right) = \rho^2 + 4\rho_1^2 = QD^2,$$

whence
$$\frac{ds'}{d\psi} = QD.$$

Also if the tangents at Q and P make angles ψ' and ψ with a fixed straight line,

$$\psi = \psi' + \frac{\pi}{2} - \phi$$

and therefore

$$\frac{d\psi'}{d\psi} = 1 + \frac{d\phi}{d\psi}. \quad \text{But } \sec^2\phi\frac{d\phi}{d\psi} = \frac{\rho_1^2 - \rho\rho_2}{2\rho_1^2},$$

therefore
$$\frac{d\psi'}{d\psi} = 1 + \frac{2(\rho_1^2 - \rho\rho_2)}{4\rho_1^2 + \rho^2}.$$

Hence
$$\rho' = \frac{ds'}{d\psi'} = \frac{ds'}{d\psi}\bigg/\frac{d\psi'}{d\psi} = \frac{(\rho^2 + 4\rho_1^2)^{\frac{3}{2}}}{\rho^2 + 6\rho_1^2 - \rho\rho_2}.$$

(3) The centre of curvature at Q may be geometrically determined as follows.

Differentiating
$$2\rho_1\sin\phi - \rho\cos\phi = 0$$

we get
$$\frac{d\phi}{d\psi} = \frac{\rho_1\cos\phi - 2\rho_2\sin\phi}{2\rho_1\cos\phi + \rho\sin\phi}.$$

Now, if EF be the diameter of curvature at E of the second evolute and H the projection of F on QD.

$$\frac{d\phi}{d\psi} = -\frac{HD}{QD} \text{ and therefore } \frac{d\psi'}{d\psi} = 1 - \frac{DH}{TD} = \frac{TH}{TD}$$

Hence $\rho' = QD^2/TH$, since $QD = TD$.

III.

More generally, let PQ be the chord of curvature subtending a constant angle 2λ at the centre, then the properties of the locus of Q may be similarly determined.

1. PQ, P′Q′ (Fig. 19) being consecutive positions, it is obvious that they intersect at N so that C, N, P, P′ are cyclic, and therefore CN is ultimately perpendicular to PN.

If PQ be projected on P′Q′, we see that, in the limit,

$$\tan\phi = \rho\sin\lambda/(2\rho_1\sin\lambda - \rho\cos\lambda)$$

(2) Let CD be the chord of curvature of the evolute subtending an angle 2λ at its centre E; then

$$\tan\phi = \frac{\text{QN}}{\text{CD} - \text{CN}} = \frac{\text{QN}}{\text{ND}} = \tan\text{QDN}.$$

Hence $\phi = \text{QDN}$; that is, the normal at Q passes through D as in case II.

(3) Also $$(\text{QQ}')^2 = (\text{Q}q)^2 + (\text{Q}'q)^2$$

$$\therefore \left(\frac{ds'}{d\psi}\right)^2 = \rho^2\sin^2\lambda + (2\rho_1\sin\lambda - \rho\cos\lambda)^2$$

$$= \rho^2 - 4\rho\rho_1\sin\lambda\cos\lambda + 4\rho_1^2\sin^2\lambda$$

But $\psi + \phi - \psi' = \lambda$ and therefore $\dfrac{d\psi'}{d\psi} = 1 + \dfrac{d\phi}{d\psi}$

$$\therefore \rho' \equiv \frac{ds'}{d\psi'} = \frac{(\rho^2 + 4\rho_1^2\sin^2\lambda - 4\rho\rho_1\sin\lambda\cos\lambda)^{\frac{3}{2}}}{\rho^2 - 4\rho\rho_1\sin\lambda\cos\lambda + 6\rho_1^2\sin^2\lambda - 2\rho\rho_2\sin^2\lambda}$$

after reduction.

(4) Again $\tan\phi = \rho\sin\lambda/(2\rho_1\sin\lambda - \rho\cos\lambda)$

may be written in the form

$$(2\rho_1\sin\lambda - \rho\cos\lambda)\sin\phi - \rho\sin\lambda\cos\phi = 0$$

Differentiating with respect to ψ, we have

$$\{(2\rho_2\sin\lambda - \rho_1\cos\lambda)\sin\phi - \rho_1\sin\lambda\cos\phi\} + \{(2\rho_1\sin\lambda - \rho\cos\lambda)\cos\phi + \rho\sin\lambda\sin\phi\}\frac{d\phi}{d\psi} = 0$$

this leads to the geometrical construction in Figure 20 where EF is the chord of curvature of the second evolute subtending an angle 2λ at its centre. Hence as in II. we find

$$\text{(i)} \qquad \frac{d\phi}{d\psi} = -\frac{\mathrm{DH}}{\mathrm{QD}},$$

$$\text{(ii)} \qquad \frac{d\psi'}{d\psi} = \frac{\mathrm{TH}}{\mathrm{TD}},$$

and $$\text{(iii)} \qquad \rho' = \mathrm{QD}^2/\mathrm{TH}.$$

IV.

If in II. or III. the tangent at Q be drawn to the circle of curvature, the envelope of this tangent is closely connected with the evolute of P.

(1) If R (Fig. 21) is the intersection of consecutive tangents RQ'QN is a circle, and therefore $\angle$ QRN is ultimately equal to $\phi = \angle$ QDN. Thus QNDR is cyclic and $\angle$ QRD is a right angle. Hence the envelope touches the tangent at the foot of the perpendicular from D, where CD is the chord of curvature of the evolute subtending the angle 2λ at the centre. Also

$$\mathrm{QR} = \mathrm{CD}\sin\lambda = 2\rho_1\sin^2\lambda.$$

(2) If R′ be a neighbouring point on this envelope

$$\begin{aligned} \mathrm{RR}' &= -\mathrm{QQ}'\cos(\phi+\lambda) + \mathrm{Q}'\mathrm{R}' - \mathrm{QR} \\ &= -\mathrm{QQ}'(\cos\phi\cos\lambda - \sin\phi\sin\lambda) + 2(\rho_1+\delta\rho_1)\sin^2\lambda - 2\rho_1\sin^2\lambda \\ &= -\cos\lambda(2\rho_1\sin\lambda - \rho\cos\lambda)\delta\psi + \rho\sin^2\lambda\delta\psi + 2\delta\rho_1\sin^2\lambda \end{aligned}$$

$$\therefore\ \rho' = \frac{ds'}{d\psi'} = 2\rho_2\sin^2\lambda + \rho - 2\rho_1\sin\lambda\cos\lambda,$$

since $\psi' - \psi = \pi - 2\lambda$ and therefore $\delta\psi' = \delta\psi$.

(3) If EF be the chord of the second evolute as in III., its projection is $\mathrm{DG} = 2\rho_2\sin^2\lambda$, and RD = sum of projections of $\mathrm{CQ}_1\mathrm{CD} = \rho - 2\rho_1\sin\lambda\cos\lambda$. Hence $\mathrm{RG} = \rho'$, that is the centre of curvature of the envelope is the projection of F on the normal at R.

V.

If on the tangent at P a length PQ equal to the radius of curvature be measured, the locus of Q has the following properties.

By projection we have (Fig. 22)

$$\tan\phi = \mathop{\mathrm{L}} \rho d\psi/(\mathrm{PQ} + \mathrm{PP'} - \mathrm{P'Q'}) = \mathop{\mathrm{L}} \rho\delta\psi/(\delta s - \delta\rho)$$
$$= \rho/(\rho - \rho_1) = \tan \mathrm{EPQ},$$

if E be the centre of curvature of the evolute. Hence the tangent at Q is the reflection of EQ in the tangent at P.

Again $$(\mathrm{QQ'})^2 = \rho^2\delta\psi^2 + (\delta s - \delta\rho)^2$$

and therefore $$\frac{ds'}{d\psi'} = \mathrm{EQ}. \quad \text{But } \psi' = \psi - \phi,$$

whence $$\rho' = \frac{ds'}{d\psi'} = \frac{ds'}{d\psi} \bigg/ \frac{d\psi'}{d\psi} = \frac{\{\rho^2 + (\rho - \rho)^2\}^{\frac{3}{2}}}{2\rho^2 - 2\rho\rho_1 + 2\rho_1{}^2 - \rho\rho_2}$$

If a length be measured on the opposite side of the tangent, we find $\tan\phi = \rho/(\rho + \rho_1)$, etc.

For any length $\kappa\rho$ measured along the tangent, similar results may be established with slight modifications.

Poles and Polars of a Conic.

By Professor J. JACK.

(*Read* 8*th November* 1907.)

(ABSTRACT.)

1. Being given a fixed line (the directrix) and two fixed points S, s (Fig. 15), then, if z and Z are two points on the directrix, the lines zp and ZP are said to correspond if zp be parallel to SZ and sz be parallel to ZP.

Theorem 1. If pairs of corresponding lines meet in p and P respectively, then sp is parallel to SP; and, if any line goes through p, the corresponding line goes through P.

Let pz, pz' correspond to PZ, PZ$'$, then sz and sz' are parallel to PZ, and PZ$'$. Hence

$$\frac{\mathrm{PZ}}{\mathrm{ZZ'}} = \frac{sz}{zz'}; \text{ and similarly } \frac{\mathrm{SZ}}{\mathrm{ZZ'}} = \frac{pz}{zz'}$$

$$\therefore \quad \frac{\mathrm{PZ}}{\mathrm{ZS}} = \frac{sz}{zp} \text{ and } \mathrm{P\hat{Z}S} = \widehat{szp}.$$

Hence Δ PZS is similar to Δ szp and PS$'$ is parallel to sp; and clearly, if p is on zp, P is on the corresponding line ZP.

Theorem 2. If a pair of points p, q, correspond to a pair P, Q, and if pq and PQ meet the directrix in z and Z, then PQ is parallel to sz and pq is parallel to SZ.

Draw SZ parallel to zp, and through Z a line parallel to sz. Then P must lie on this line, Q therefore also lies on it, and therefore PQ is parallel to sz; and so on.

Theorem 3. If the point P moves in a conic with S as focus and the given line as directrix, p traces out a circle.

For, making PZ perpendicular to the directrix for convenience simply, we have

$$\frac{\mathrm{SP}}{\mathrm{PZ}} = e = \frac{sp}{sz} \quad \therefore \; sp = e.sz = \text{constant.}$$

The circle is an eccentric circle of the conic.

2. Parallel lines on the one system correspond to lines meeting on the directrix in the other. Hence, propositions like the following can be proved.

If pairs of points be taken on three concurrent lines, the three points of intersection of lines joining pairs of corresponding points are collinear.

For the three concurrent lines can be transformed into three parallel lines and the pairs of points into pairs of points on parallel lines, a particular case in which the theorem is easily proved.

Tangents to a curve in one system correspond to tangents in the other, and chords of contact to chords of contact. Whence the usual theorems regarding tangents to a conic are easily proved.

The above transformation is easily seen to be a particular case of the projective transformation, its analytical representation being of the form

$$x = k/X, \; y = -lY/X.$$

3. The following is an example of the application of the method in the proof of theorems regarding poles and polars (Fig. 16).

If QQ′, a chord of a conic with S as focus and the given line as directrix, passes through a fixed point O, PQ and PQ′, the tangents at Q and Q′ meet in P, which lies on a fixed straight line.

Let QQ′ meet the directrix in Z. Take any point s and draw sz parallel to QZ, zq parallel to SZ, sq parallel to SQ and sq' parallel SQ′. Then q and q' are on the eccentric circle of s. Draw so parallel to SO and let kp, the polar of o, cut the directrix in k. Finally draw sp parallel to SP.

Since sp bisects qsq', the tangents at q and q' meet in sp, and, since kp is the polar of o, the tangents meet on kp. Therefore pq and pq' are tangent to the eccentric circle, and they correspond to the lines PQ, PQ′. Hence p and k correspond to P and K and therefore KP is parallel to sk. Now K is a fixed point and KP is parallel to a fixed line. Hence the proposition is proved.

All the theorems regarding poles and polars to a conic may be proved in a similar manner.

A Problem in the Theory of Numbers.

By T. H. Miller.

(*Read* 12*th June*. *MS. received* 19*th January* 1908.)

One of the well known properties of the number 7 is that when $\frac{1}{7}$ is reduced to a decimal, the periods of two digits are obtained to infinity by successive doubling. It is interesting to find for what other numbers this property is true.

Let n be any number, and r the base of notation, then

$$\frac{1}{n} = \frac{2n}{r^2} + \frac{4n}{r^4} + \frac{8n}{r^6} + \text{ to infinity}$$

$$= \frac{2n}{r^2 - 2}$$

Therefore $r^2 = 2n^2 + 2$ where r and n are integers.

To find a solution of this equation, write it in the form

$$r^2 - 2n^2 = 2$$

$$\therefore\ (r + \sqrt{2}.n)(r - \sqrt{2}.n) = (2 - \sqrt{2})(2 + \sqrt{2})\{(1 + \sqrt{2})(1 - \sqrt{2}\}^{2p}$$

where p is any integer

$$\therefore\ (r + \sqrt{2}.n)(r - \sqrt{2}.n)$$

$$= \left[\left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\} + \left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\}\right]$$

$$\times \left[\left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\} - \left\{\frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p}\right\}\right]$$

Now, from the symmetry of the expression, this equation is satisfied if we make

$$r = \frac{2-\sqrt{2}}{2}(1+\sqrt{2})^{2p} + \frac{2+\sqrt{2}}{2}(1-\sqrt{2})^{2p},$$

$$n = \frac{2-\sqrt{2}}{2\sqrt{2}}(1+\sqrt{2})^{2p} - \frac{2+\sqrt{2}}{2\sqrt{2}}(1-\sqrt{2})^{2p};$$

both expressions being integral. Now if

$$(2 - \sqrt{2})(1 + \sqrt{2})^{2p} = M_p + N_p\sqrt{2},$$

then $r_p = M_p$ and $n_p = N_p$, where p may have any positive integral value.

Expanding, and writing $\overset{p}{C}_k$ for the number of combinations of p things taken k together

$$r_p = (2p-1)2 + \overset{2p-1}{C_3} . 4 + \overset{2p-1}{C_5} . 8 + \overset{2p-1}{C_7} . 2^4 + \dots + (2p-1)\,2^{p-1} + 2^p.$$

and $$n_p = 1 + \overset{2p-1}{C_2} . 2 + \overset{2p-1}{C_4} . 2^2 + \dots \qquad + \overset{2p-1}{C_2} . 2^{p-2} + 2^{p-1} .$$

On substituting for p in succession 1, 2, 3, etc., we get

$$\begin{aligned} r_1 &= 2, & n_1 &= 1. \\ r_2 &= 10, & n_2 &= 7. \\ r_3 &= 58, & n_3 &= 41. \\ r_4 &= 338, & n_4 &= 239. \\ r_5 &= 1970, & n_5 &= 1393, \text{ etc.} \end{aligned}$$

The first is an obvious illustration, as

$$\frac{2}{2^2} + \frac{4}{2^4} + \frac{8}{2^6} + \text{to infinity} = \frac{1}{1}.$$

That the third number has the same property, can be proved from the infinite geometrical progression, or may be tested by reducing $\frac{1}{41}$ to a radix fraction in scale 58.

If we adopt the following notation for numbers: 0 to 9 to be expressed as usual by arabic figures, ten to nineteen by t, t_1, t_2 .. t_9, twenty by T, thirty θ, forty f, fifty F, with subscript figures for the excess above multiples of ten; then f_1 represents forty-one, and

$$\frac{1}{f_1} = \dot{1}T_4 2 f_8 5 \theta_8 t_1 t_8 T_2 \theta_6 f_5 t_5 \theta_2 \theta_1 7 4 t_4 8 T_8 t_6 F_6 \theta_3 F_5 9 F_2 t_9 f_6 \theta_9 \theta_5 T_1 t_2 f_2 T_5 T_6 F F_3 f_3 f_4 T_9 \dot{f_1}$$

Now $2 \times f_1 = 1T_4$; $2 \times 1T_4 = 2f_8$, etc.

The number n and the base r are connected by various relations, such as

$$\begin{aligned} n_p &= 2r_{p-1} + 3n_{p-1}, \\ r_p - r_{p-1} &= n_{p-1} + n_p, \\ r_p - 6r_{p-1} + r_{p-2} &= 0, \end{aligned}$$

from which other forms of the series for n and r can be obtained and some properties of n proved. Thus, the number of recurring figures in $\frac{1}{n}$ is always $n-1$ and the last figure is always n.

Edinburgh Mathematical Society.

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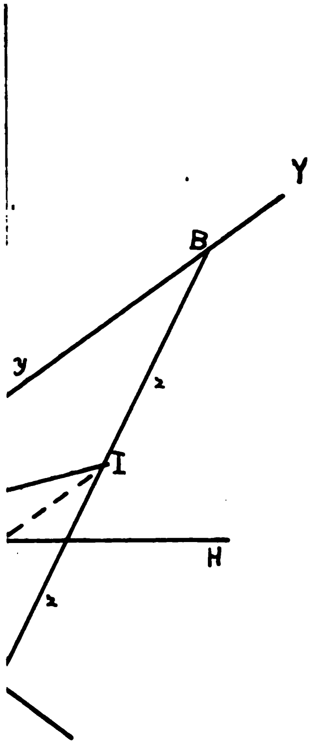

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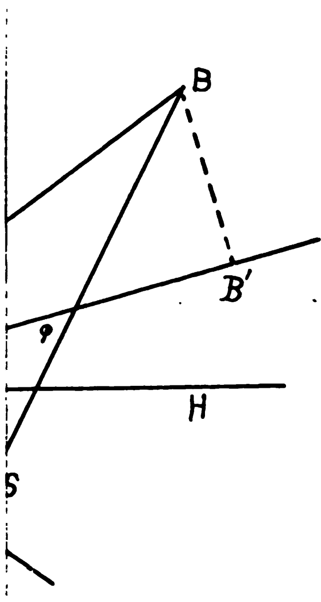

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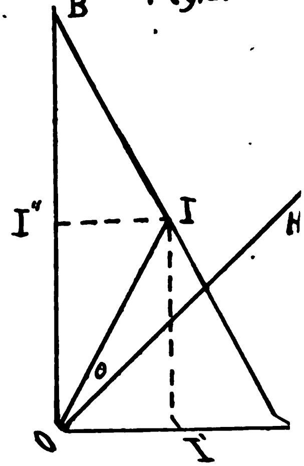

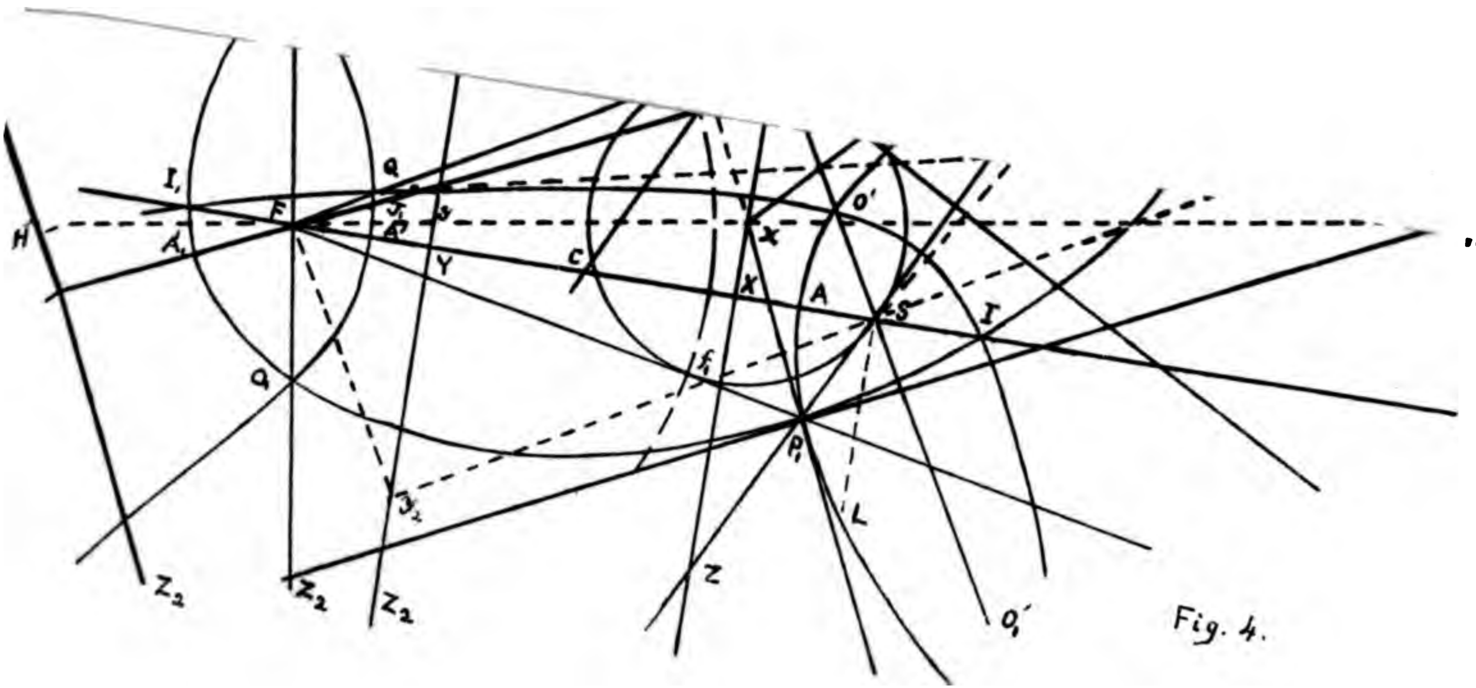

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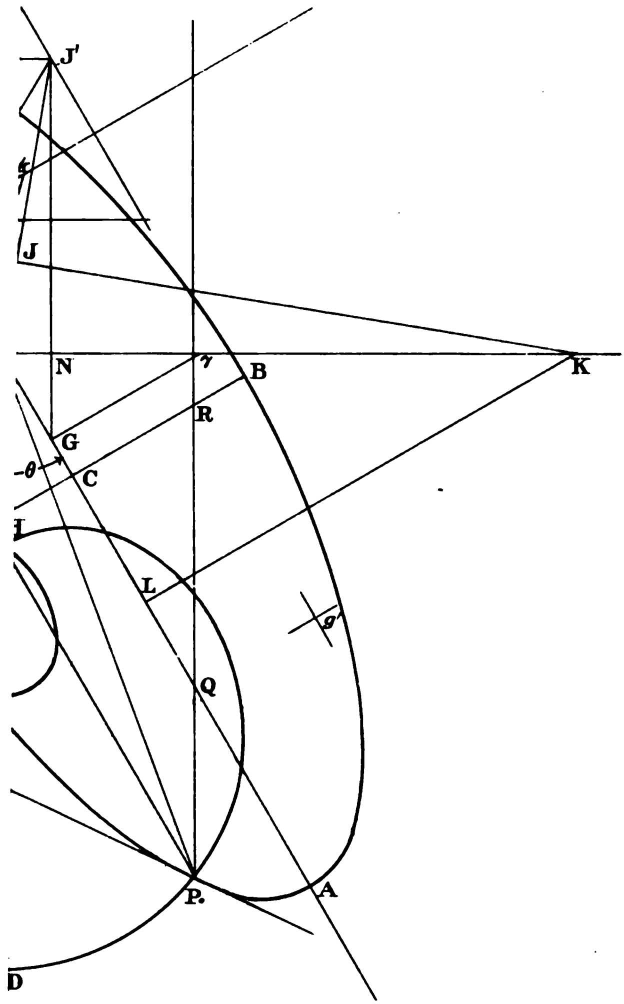

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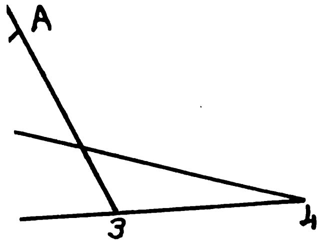

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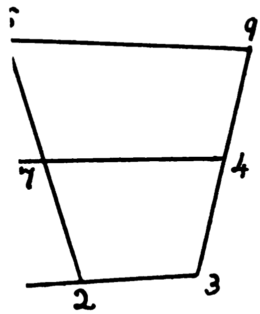

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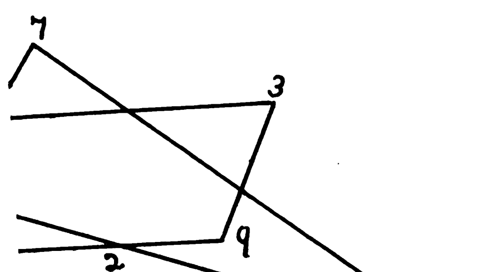

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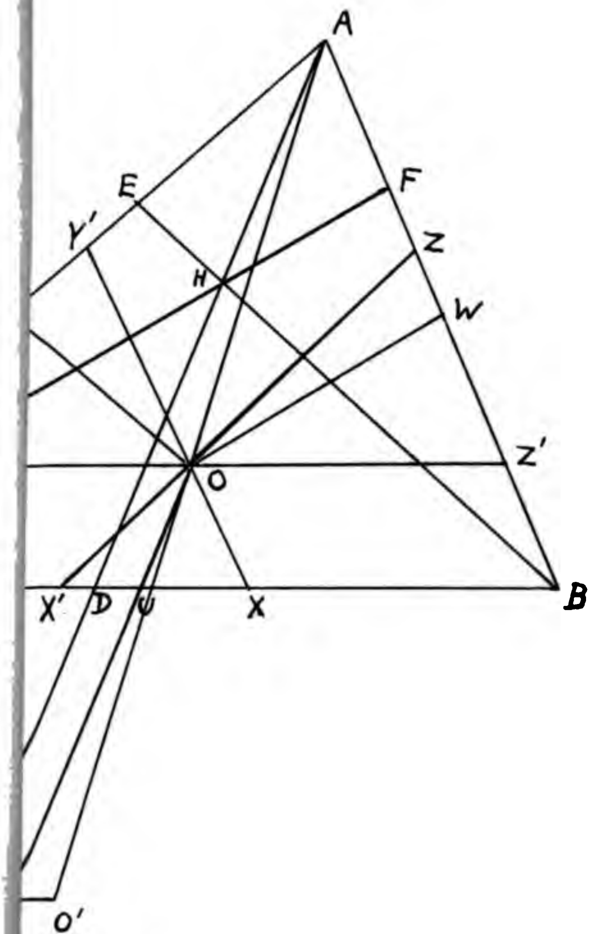

Fig. 10.

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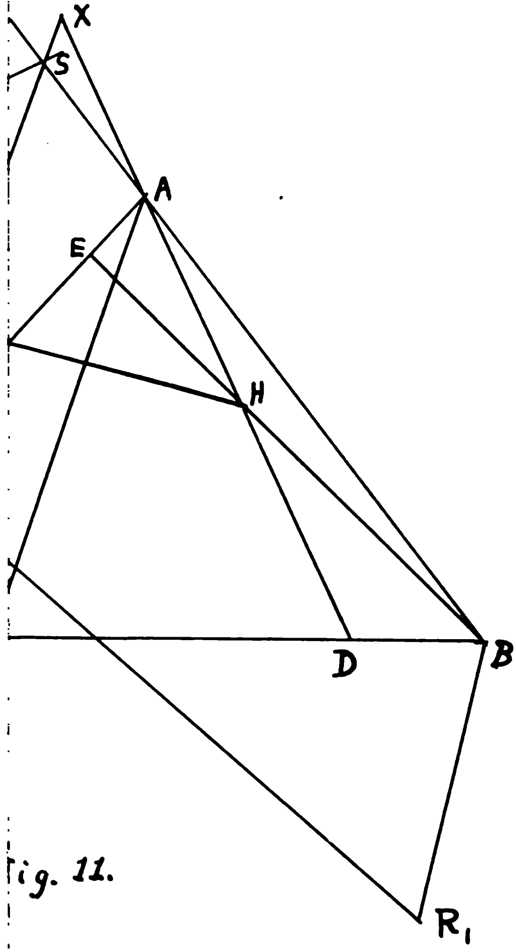

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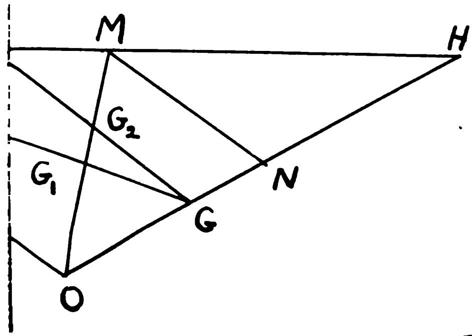

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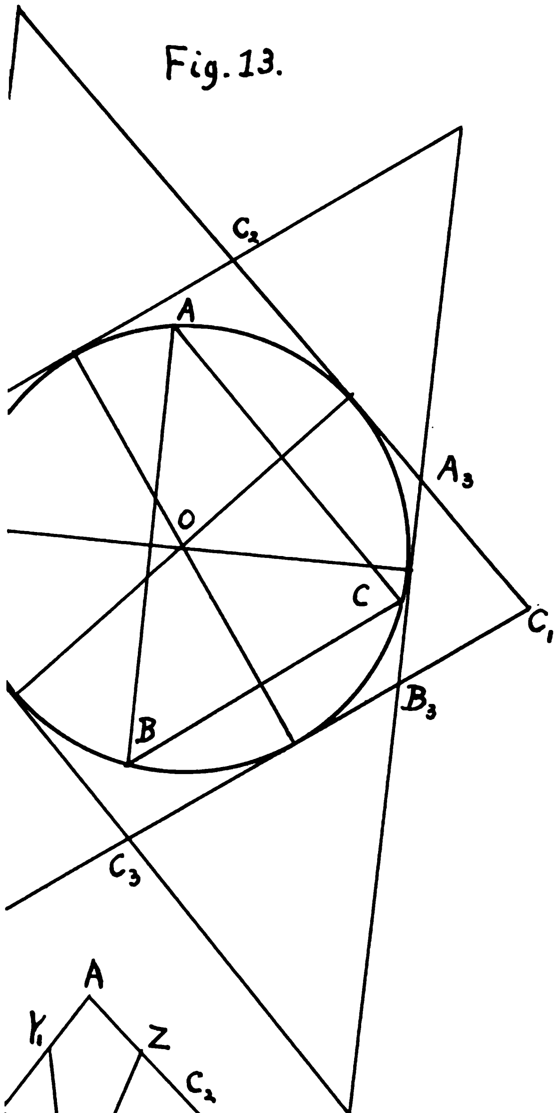

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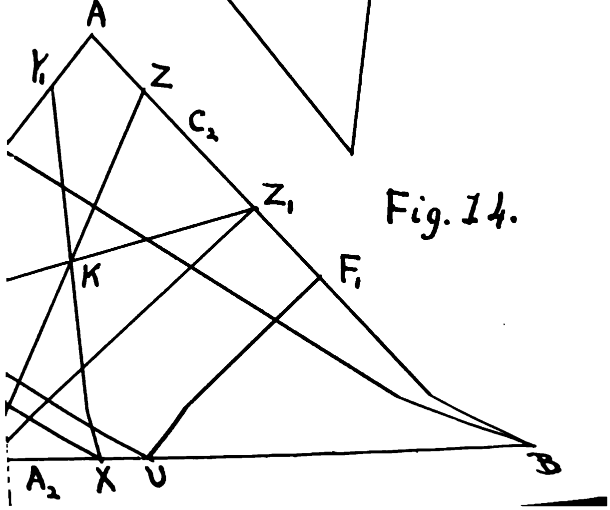

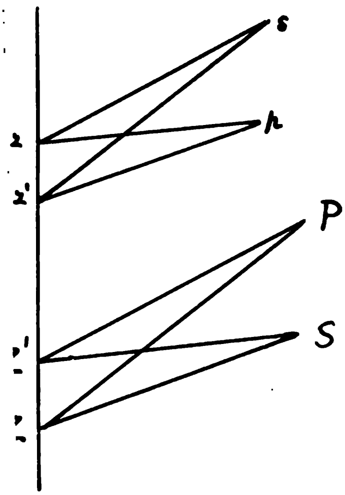
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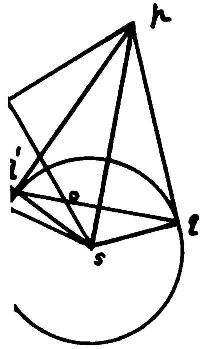
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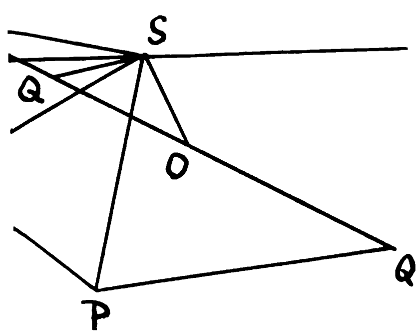
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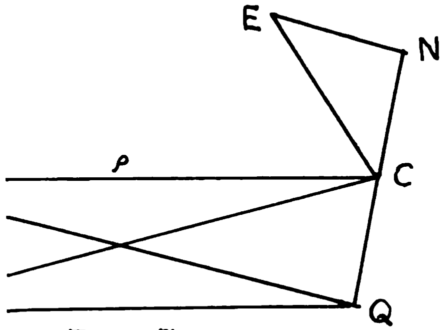

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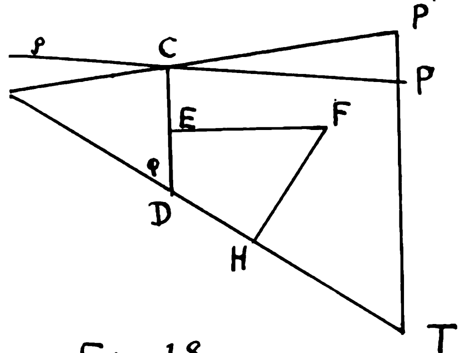

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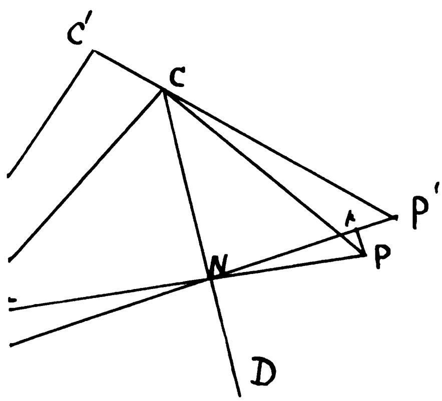

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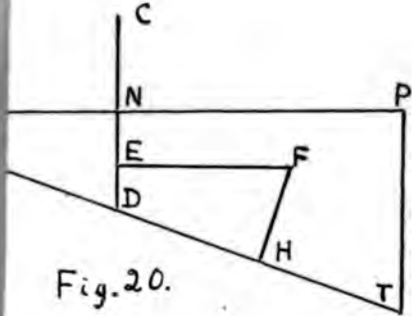

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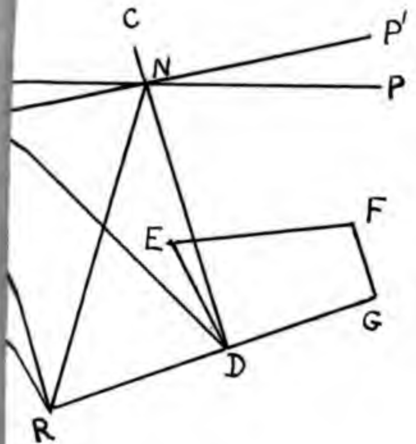

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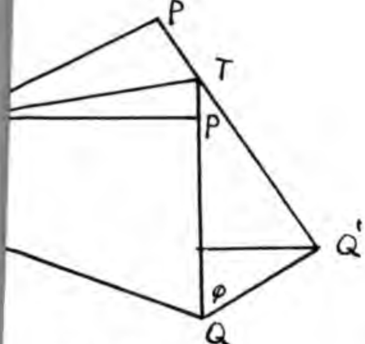

Fig. 22.